SF2524 Matrix Computations for Large-scale Systems
Exam

Aids: None **Time: Four hours**

Grades: E: 16 points, D: 19 points, C: 22 points, B: 25 points, A: 28 points (out of the possible 35 points, including bonus points from homeworks).

Problem 1 (5p) Consider the linear system of equations $Ax = b$. The min-max bound for GMRES states that

$$\|Ax_n - b\| \leq \alpha \min_{p \in P_n^0} \max_{\lambda \in \lambda(A)} |p(\lambda)|$$

where α is independent of n.

(a) Suppose A is diagonalizable and the eigenvalues of A are contained in a disk of radius $\rho > 0$ centered at $c \in \mathbb{C}$ and $|c| > \rho$. Derive a formula for a constant $\beta < 1$ such that $\|Ax_n - b\| \leq \alpha\beta^n$ for all $n > 0$.

(b) The modified linear system of equations $\tilde{A}z = \tilde{b}$, where $\tilde{A} = \gamma A$ and $\tilde{b} = \gamma b$, has the same solution as $Ax = b$ for any $\gamma \neq 0$, since $x = A^{-1}b = \tilde{A}^{-1}\tilde{b} = z$. Show that GMRES applied to $\tilde{A}z = \tilde{b}$ generates the same sequence of approximations as GMRES applied to $Ax = b$.

Problem 2 (4p) Consider a diagonalizable matrix $A \in \mathbb{R}^{(2N+1)\times(2N+1)}$ with eigenvalues

$$\begin{aligned}
\lambda_1 &= 1 + \varepsilon \\
\lambda_{2k} &= 2 + \sin(2k\pi/N), \ \ k = 1, \ldots, N \\
\lambda_{2k+1} &= 2 + i\cos(2k\pi/N), \ \ k = 1, \ldots, N
\end{aligned}$$

(a) Suppose $\varepsilon = -0.5$. Let Q_k be the orthogonal matrix generated by k steps of Arnoldi's method. Derive a constant α such that the indicator $\|(I - Q_kQ_k^T)x_1)\|_2$ can be bounded by

$$\|(I - Q_kQ_k^T)x_1)\|_2 < \xi\alpha^k,$$

for some value $\xi > 0$, where x_1 is the eigenvector associated with eigenvalue λ_1. You do not have to specify the constant ξ and you may directly use theorems from the course.

(b) Suppose $N = 100$. To which eigenvalue will the power method converge if $\varepsilon = -2$, $\varepsilon = -0.5$, $\varepsilon = 1$, $\varepsilon = 3$? Assume that the starting vector is such that it has components in all eigenvector directions.

Problem 3 (5p) Let $f(A) = A^{1/3}$. Consider the iteration

$$X_{k+1} = \alpha X_k + \beta X_k^{-1} + \gamma X_k^{-2}$$

for $k = 1, \ldots,$ and $X_0 = A$ where A is symmetric positive definite. Derive constants α, β and γ such that (if the iteration converges) it converges quadratically to $f(A)$. Justify the quadratic convergence.

Problem 4 (3p) Let $\|z\|_B := \sqrt{z^T B z}$. Consider the linear system of equations $Ax_* = b$, where $A \in \mathbb{R}^{m\times m}$ is symmetric positive definite.

(a) Let x be an approximation of x_*. Show that $\|x - x_*\|_A = \|Ax - b\|_{A^{-1}}$.

(b) Let e_n be the error in step n of CG. Show that $\|e_{n+1}\|_A \leq \|e_n\|_A$ using the fact that the CG-iterates minimize the error in the $\|\cdot\|_A$-norm over an associated Krylov subspace.

(c) Let e_n be the error in step n of CG. A theorem in this course stated that

$$\frac{\|e_n\|_A}{\|e_0\|_A} \leq \min_{p\in P_n^0} \max_{\lambda\in\lambda(A)} |p(\lambda)|.$$

Suppose all the eigenvalues are explicitly $\lambda_1 = 10$ and $\lambda_k = 2 + 1/k$ for $k = 2, \ldots, 100$. What is maximum (worst-case) error $\|e_n\|_A$ after 100 iterations?

Problem 5 (3p) Let $Q = (q_1, \ldots, q_n) \in \mathbb{R}^{m\times n}$ be an orthogonal matrix. Suppose $b \in \mathbb{R}$ is such that $b \notin \operatorname{span}(q_1, \ldots, q_m)$.

(a) Derive the Gram-Schmidt procedure by computing explicit formulas for $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ and $q_{n+1} \in \mathbb{R}^m$ such that $Q^T q_{n+1} = 0$, $\|q_{n+1}\| = 1$, $\operatorname{span}(q_1, \ldots, q_{n+1}) = \operatorname{span}(q_1, \ldots, q_n, b)$ and

$$b = h_1 q_1 + \ldots + h_n q_n + \gamma q_{n+1}.$$

Express the procedure using only products of matrices and vectors (no for-loops).

(b) Describe the double Gram-Schmidt procedure (any version). What are the advantages and disadvantages of classical Gram-Schmidt and double Gram-Schmidt?

Problem 6 (3p) Let

$$A = \begin{pmatrix} \alpha & \times & \times \\ \beta & \times & \times \\ \gamma & \times & \times \end{pmatrix}$$

Derive a formula (involving α, β and γ) for an orthogonal matrix Q such that QAQ^T has the structure

$$QAQ^T = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}$$

Problem 7 (6p) A matrix is called symmetric if $A^T = A$ and anti-symmetric if $A^T = -A$. Suppose more generally that $A \in \mathbb{R}^{m\times m}$ satisfies $A^T = \alpha A$ for some value $\alpha \neq 0$. Let $Q_{n+1} = (q_1, \ldots, q_{n+1}) \in \mathbb{R}^{m\times(n+1)}$ and $\underline{H}_n \in \mathbb{R}^{(n+1)\times n}$ correspond to an Arnoldi factorization for A and let $H_n \in \mathbb{R}^{n\times n}$ be the top block of $\underline{H}_n$.

(a) Show that $H_n^T = \alpha H_n$. Specify which elements of H_n will be zero (for any starting vector and any A satisfying $A^T = \alpha A$). Separate between the two cases $\alpha = 1$ and $\alpha \neq 1$.

(b) Show that for any $k > 1$ there exist c_{k-1}, a_k and b_{k+1} such that

$$Aq_k = c_{k-1} q_{k-1} + a_k q_k + b_{k+1} q_{k+1}.$$

(c) Derive a generalization of the Lanczos procedure for matrices satisfying $A^T = \alpha A$ by deriving formulas for the Arnoldi factorization corresponding to Q_{n+2}, $\underline{H}_{n+1}$ expressed in terms of the Arnoldi factorization corresponding to Q_{n+1}, $\underline{H}_n$. The procedure should not be more computationally expensive than one step of the Lanczos procedure. That is, it should only involve only linear combinations of (at most) three vectors and the computation of corresponding scalar products and norms.