Reachability and Controllability

Forward

$$\Sigma \begin{cases} x(t+1) = Ax(t) + Bw(t) \\ y(t) = Cx(t) + Dw(t) \end{cases}$$

$$\begin{tikzcd} H^-(w) \arrow[rr, "\mathcal{H}_-"] \arrow[dr, "\mathcal{R}"'] & & H^+ \\ & X \arrow[ur, "\mathcal{O}"'] & \end{tikzcd}$$

$$\mathcal{H}_- = E^{H^+}|_{H^-(w)} = \mathcal{O}\mathcal{R}$$

where

$\mathcal{O} = E^{H^+}|_X$ is the observability op.

$\mathcal{R} = E^X|_{H^-(w)}$ is the reachability op.

$$X = \overline{\operatorname{Im}\mathcal{R}} \oplus \ker \mathcal{R}^*$$

Def: X is reachable if $\ker \mathcal{R}^* = 0$ ($\mathcal{R}$ is onto)

factorization (deterministic)

$$\begin{tikzcd} & S \perp \bar{S} \mid X \arrow[dl] \arrow[dr] & \\ H^-(w) \perp H^+ \mid X & & H^+(\bar{w}) \perp H^- \mid X \end{tikzcd}$$

Backward

$$\bar{\Sigma} \begin{cases} \bar{x}(t-1) = \bar{A}\bar{x}(t) + \bar{B}\bar{w}(t) \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{w}(t) \end{cases}$$

$$\begin{tikzcd} H^+(\bar{w}) \arrow[rr, "\mathcal{H}_+"] \arrow[dr, "\mathcal{K}"'] & & H^- \\ & X \arrow[ur, "\mathcal{C}"'] & \end{tikzcd}$$

$$\mathcal{H}_+ \doteq E^{H^-}|_{H^+(\bar{w})} = \mathcal{C}\mathcal{K}$$

where

$\mathcal{C} = E^{H^-}|_X$ is the constructibility op.

$\mathcal{K} = E^X|_{H^+(\bar{w})}$ is the controllability op.

$$X = \overline{\operatorname{Im}\mathcal{K}} \oplus \ker \mathcal{K}^*$$

X is controllable if $\ker \mathcal{K}^* = 0$ ($\mathcal{K}$ is onto)

Proposition 8.4.1: Let $(\mathbb{H}, U, X)$ be a Markovian representation with $X \sim (S, \bar{S})$, and let $S_{-\infty}$ and $\bar{S}_\infty$ be the remote past of S and the remote future of $\bar{S}$ respectively.
Then, X is reachable iff $X \cap S_{-\infty} = 0$
and controllable iff $X \cap \bar{S}_\infty = 0$.

If $(\mathbb{H}, U, X)$ is proper, X is both reachable and controllable.

Def: A Markovian representation $(\mathbb{H}, U, X)$ will be called purely nondeterministic if $X \cap S_{-\infty} = 0$ and $X \cap \bar{S}_\infty = 0$.

Proposition 8.4.2: Let Σ and $\bar{\Sigma}$ be forward and backward realizations of a finite-dimensional Markovian representation $(\mathbb{H}, U, X)$.
Then, X is reachable iff (A, B) is reachable
and X is controllable iff $(A', \bar{B})$ is reachable.

<u>Proposition 8.4.3</u>: Let $(\mathbb{H}, U, X)$ be a finite-dimensional Markovian representation of y. Then the following conditions are equivalent.

(i) X is proper.

(ii) X is reachable

(iii) X is controllable

(iv) A is a stability matrix, i.e. $|\lambda(A)| < 1$.

This can happen only if y is purely nondeterministic in both directions

i.e. $\bigcap_t U^t H^- = \bigcap_t U^t H^+ = 0$.

Wold decomposition (Cor. 4.5.9) $\Rightarrow$ $y(t) = y_0(t) + y_\infty(t) \quad t \in \mathbb{Z}$

where y_0 is purely nondeterministic in the forward direction

y_∞ is purely deterministic —— " ——

Finite-dimensional Markovian repr. $\Rightarrow$ y is reversible, i.e. $H_{-\infty}(y) = H_{+\infty}(y)$

$\Rightarrow$ y_0 is pnd also backward and y_∞ is pd backward.

<u>Theorem 8.4.5</u>: Let $(\mathbb{H}, U, X)$ be a finitedimensional Markovian representation of y with $X \sim (S, \bar{S})$ and with generating processes w and $\bar{w}$.

Then $S_{-\infty} = \bar{S}_\infty$ and $X = X_0 \oplus X_\infty$, $\mathbb{H} = \mathbb{H}_0 \oplus X_\infty$

where $X_0 \subset \mathbb{H}_0$ and $X_\infty = S_{-\infty} = \bar{S}_\infty$, $\mathbb{H}_0 = H(w) = H(\bar{w})$

are doubly invariant.

In particular, $p = \bar{p}$, i.e. w and $\bar{w}$ have the same dimension.

Moreover, $(\mathbb{H}, U_0, X_0)$ is a proper Markovian representation for y_0,

where $U_0 \doteq U|_{\mathbb{H}_0}$, and has the same generating processes as $(\mathbb{H}, U, X)$.

We call X_0 – the proper subspace of X

X_∞ – the deterministic subspace of X.

Let $Y_\infty = H_{-\infty}(y) = H_{+\infty}(y)$.

<u>Cor. 8.4.6</u>: If the process y has a finite-dimensional Markovian representation $(\mathbb{H}, U, X)$

Then $Y_\infty \subset X_\infty$. If X is observable or constructible,

then $Y_\infty = X_\infty$.

Cor 8.4.7: Let $(\mathbb{H}, U, X)$ be a finite-dimensional Markovian repr.

Then $X_0 = \operatorname{Im} \mathcal{R} = \operatorname{Im} \mathcal{K}$

and $X_\infty = \ker \mathcal{R}^* = \ker \mathcal{K}^*$

$$X = \operatorname{Im} \mathcal{R} \oplus \ker \mathcal{R}^*$$
$$X = \operatorname{Im} \mathcal{K} \oplus \ker \mathcal{K}^*$$

Determine $\Sigma, \bar{\Sigma}$ in a basis adapted to the decomposition $X = X_0 \oplus X_\infty$.

Let $\{\xi_1, \dots, \xi_{n_0}\}$ be a basis in X_0 and $\{\xi_{n_0+1}, \dots, \xi_n\}$ be a basis in X_∞.

(the dual basis is also adapted to the decomposition).

$$\Rightarrow \quad P = E\{x(t)x(t)'\} = \begin{bmatrix} P_0 & 0 \\ 0 & P_\infty \end{bmatrix} \quad \text{and} \quad \bar{P} = E\{\bar{x}(t)\bar{x}(t)'\} = \begin{bmatrix} P_0^{-1} & 0 \\ 0 & P_\infty^{-1} \end{bmatrix}.$$

Theorem 8.4.8: Let $(\mathbb{H}, U, X)$ be a finite-dim. Markovian repr. of y and let Σ and $\bar{\Sigma}$ be a dual pair of stochastic realizations with bases adapted to the decomp.

Then
$$\Sigma: \begin{cases} \begin{bmatrix} x_0(t+1) \\ x_\infty(t+1) \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A_\infty \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_\infty(t) \end{bmatrix} + \begin{bmatrix} B_0 \\ 0 \end{bmatrix} w(t) \\ y(t) = \begin{bmatrix} C_0 & C_\infty \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_\infty(t) \end{bmatrix} + D w(t). \end{cases}$$

where $|\lambda(A_0)| < 1$, $|\lambda(A_\infty)| = 1$, (A_0, B_0) is reachable

and $\operatorname{Im} \mathcal{R} = \{a' x_0(0) \mid a \in \mathbb{R}^{n_0}\}$ $\quad \ker \mathcal{R}^* = \{a' x_\infty(0) \mid a \in \mathbb{R}^{n-n_0}\}$

Moreover, $y_0(t) = C x_0(t) + D w(t)$ is the p.n.d. part of y

and $y_\infty(t) = C_\infty x_\infty(t)$ is the p.d. —— 〃 ——.

Dually,
$$\bar{\Sigma}: \begin{cases} \begin{bmatrix} \bar{x}_0(t-1) \\ \bar{x}_\infty(t-1) \end{bmatrix} = \begin{bmatrix} A_0' & 0 \\ 0 & A_\infty' \end{bmatrix} \begin{bmatrix} \bar{x}_0(t) \\ \bar{x}_\infty(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_0 \\ 0 \end{bmatrix} \bar{w}(t) \\ y(t) = \begin{bmatrix} \bar{C}_0 & \bar{C}_\infty \end{bmatrix} \begin{bmatrix} \bar{x}_0(t) \\ \bar{x}_\infty(t) \end{bmatrix} + \bar{D} \bar{w}(t) \end{cases}$$

where $(A_0', \bar{B}_0)$ is reachable

and $\operatorname{Im} \mathcal{K} = \{a' \bar{x}_0(-1) \mid a \in \mathbb{R}^{n_0}\}$ $\quad \ker \mathcal{K}^* = \{a' \bar{x}_\infty(-1) \mid a \in \mathbb{R}^{n-n_0}\}$.

Finally, $\bar{C}_0 = C_0 P_0 A_0' + D B_0'$ $\quad \bar{C}_\infty = C_\infty P_\infty A_\infty'$

and $X_\infty = Y_\infty$ iff (C_∞, A_∞) is observable, or, equivalently iff $(\bar{C}_\infty, A_\infty')$ is observable.

Minimality and non-minimality of finite-dimensional models.

<u>Proposition 8.5.1</u>: Let Σ be the linear stochastic system $\begin{cases} x(t+1) = Ax(t) + Bw(t) \\ y(t) = Cx(t) + Dw(t) \end{cases}$ and $W(z) = C(zI-A)^{-1}B + D$ its transfer fcn.

Let $\Phi(z) \triangleq W(z)W(z^{-1})'$ and $X = \{a'x(0) \mid a \in \mathbb{R}^n\} = X_0 \oplus X_\infty$ where X_∞ is the deterministic subspace of X.

Then $\Phi(e^{i\theta})$ is the spectral density of the p.n.d part of y

and $\frac{1}{2}\deg\Phi \le \deg W \le \dim X_0 \le \dim X \le \dim\Sigma$.

Moreover

(i) $\frac{1}{2}\deg\Phi = \deg W$ iff W is a minimal spectral factor

(ii) $\deg W = \dim X_0$ iff (C,A) is observable.

(iii) $\dim X_0 = \dim X$ iff $|\lambda(A)| < 1$.

(iv) $\dim X = \dim\Sigma$ iff $x(0)$ is a basis in X

(v) $\dim X_0 = \dim\Sigma$ iff (A,B) is reachable.

In particular, if y is p.n.d., then Σ is a minimal stochastic realization of y iff (i), (ii) and (v) hold.

Otherwise, Σ is minimal iff (i), (ii) and (iv) hold.

<u>Proposition 8.5.2</u>: An observable system Σ with A a stability matrix is a minimal realization of y iff its steady state Kalman filter

$$\begin{cases} \hat{x}_\infty(t+1) = A\hat{x}_\infty(t) + K_\infty v(t) \\ y(t) = C\hat{x}_\infty(t) + G v(t) \end{cases}$$

is completely reachable.

(In the sense that (A, K_∞) is reachable).