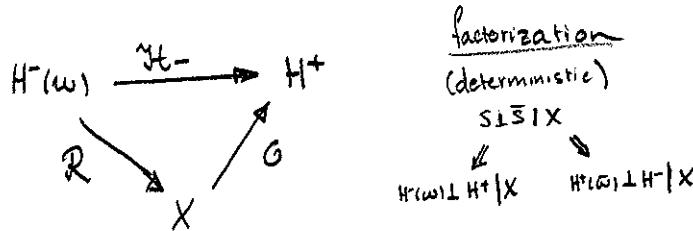


Reachability and Controllability

Forward

$$\Sigma \left\{ \begin{array}{l} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array} \right.$$



$$H_- = E^{H^+}_{|H^-(w)} = GR$$

where

$G = E^{H^+}_{|X}$ is the observability op.

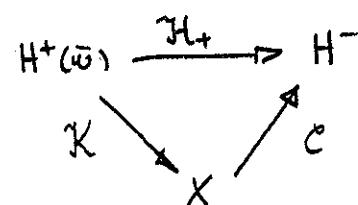
$R = E^X_{|H^-(w)}$ is the reachability op.

$$X = \overline{\text{Im } R} \oplus \ker R^*$$

Def: X is reachable if $\ker R^* = 0$ (R is onto)

Backward

$$\Xi \left\{ \begin{array}{l} \bar{x}(t-1) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t) \end{array} \right.$$



$$H_+ = E^H_{|H^+(w)} = CK$$

where

$C = E^H_{|X}$ is the constructibility op.

$K = E^X_{|H^+(w)}$ is the controllability op.

$$X = \overline{\text{Im } K} \oplus \ker K^*$$

X is controllable if $\ker K^* = 0$ (K is onto)

Proposition 8.4.1: Let (H, U, X) be a Markovian representation with $X \sim (S, \bar{S})$, and let $S_{-\infty}$ and \bar{S}_{∞} be the remote past of S and the remote future of \bar{S} respectively. Then, X is reachable iff $X \cap S_{-\infty} = 0$ and controllable iff $X \cap \bar{S}_{\infty} = 0$.

If (H, U, X) is proper, X is both reachable and controllable.

Def: A Markovian representation (H, U, X) will be called purely nondeterministic if $X \cap S_{-\infty} = 0$ and $X \cap \bar{S}_{\infty} = 0$.

Proposition 8.4.2: Let Σ and Ξ be forward and backward realizations of a finite-dimensional Markovian representation (H, U, X) .

Then, X is teachable iff (A, B) is reachable and X is controllable iff (A', \bar{B}) is reachable.

Proposition 8.4.3: Let $(\mathbb{H}, \mathcal{U}, X)$ be a finite-dimensional Markovian representation of y . Then the following conditions are equivalent.

- (i) X is proper.
- (ii) X is reachable
- (iii) X is controllable
- (iv) A is a stability matrix, i.e. $|\lambda(A)| < 1$.

This can happen only if y is purely nondeterministic in both directions

$$\text{i.e. } \Lambda_t U^t H^- = \Lambda_t U^t H^+ = 0.$$

Wold decomposition (Cor. 4.5.9) $\Rightarrow y(t) = y_0(t) + y_{\infty}(t) \quad t \in \mathbb{Z}$

where y_0 is purely nondeterministic in the forward direction

y_0 is purely deterministic \dashv

Finite-dimensional Markovian repr. $\Rightarrow y$ is reversible, i.e. $H_{-\infty}(y) = H_{+\infty}(y)$
 $\Rightarrow y_0$ is pd also backward and y_{∞} is pd backward.

Theorem 8.4.5: Let $(\mathbb{H}, \mathcal{U}, X)$ be a finitedimensional Markovian representation of y with $X \sim (S, \bar{S})$ and with generating processes w and \bar{w} .

Then $S_{-\infty} = \bar{S}_{\infty}$ and $X = X_0 \oplus X_{\infty}$, $\mathbb{H} = \mathbb{H}|_{X_0} \oplus X_{\infty}$

where $X_0 \subset \mathbb{H}|_{X_0}$ and $X_{\infty} = S_{-\infty} = \bar{S}_{\infty}$, $\mathbb{H}|_{X_0} = H(w) = H(\bar{w})$
 are doubly invariant.

In particular, $p = \bar{p}$, i.e. w and \bar{w} have the same dimension.

Moreover, $(\mathbb{H}|_{X_0}, \mathcal{U}_0, X_0)$ is a proper Markovian representation for y_0 ,
 where $\mathcal{U}_0 \stackrel{\cong}{=} \mathcal{U}|_{\mathbb{H}|_{X_0}}$, and has the same generating processes as $(\mathbb{H}, \mathcal{U}, X)$.

We call X_0 - the proper subspace of X

X_{∞} - the deterministic subspace of X .

Let $Y_{\infty} = H_{-\infty}(y) = H_{+\infty}(y)$.

Cor. 8.4.6: If the process y has a finite-dimensional Markovian representation $(\mathbb{H}, \mathcal{U}, X)$

Then $Y_{\infty} \subset X_{\infty}$. If X is observable or constructible,
 then $Y_{\infty} = X_{\infty}$.

Cor 8.4.7: Let $(\mathbb{H}, \mathcal{U}, X)$ be a finite-dimensional Markovian repr.

$$\text{Then } X_0 = \text{Im } R = \text{Im } K$$

$$\text{and } X_{\infty} = \ker R^* = \ker K^*$$

$$X = \text{Im } R \oplus \ker R^*$$

$$X = \text{Im } K \oplus \ker K^*$$

Determine $\Sigma, \bar{\Sigma}$ in a basis adapted to the decomposition $X = X_0 \oplus X_{\infty}$.

Let $\{\xi_1, \dots, \xi_{n_0}\}$ be a basis in X_0 and $\{\xi_{n_0+1}, \dots, \xi_n\}$ be a basis in X_{∞} .

(the dual basis is also adapted to the decomposition).

$$\Rightarrow P = E\{X(t)X(t)'\} = \begin{bmatrix} P_0 & 0 \\ 0 & P_{\infty} \end{bmatrix} \quad \text{and} \quad \bar{P} = E\{\bar{X}(t)\bar{X}(t)'\} = \begin{bmatrix} P_0^{-1} & 0 \\ 0 & P_{\infty}^{-1} \end{bmatrix}$$

Theorem 8.4.8: Let $(\mathbb{H}, \mathcal{U}, X)$ be a finite-dim. Markovian repr. of y and let Σ and $\bar{\Sigma}$ be a dual pair of stochastic realizations with bases adapted to the decomp.

$$\text{Then } \Sigma : \left\{ \begin{bmatrix} X_0(t+1) \\ X_{\infty}(t+1) \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A_{\infty} \end{bmatrix} \begin{bmatrix} X_0(t) \\ X_{\infty}(t) \end{bmatrix} + \begin{bmatrix} B_0 \\ 0 \end{bmatrix} w(t) \right. \\ \left. y(t) = [C_0 \ C_{\infty}] \begin{bmatrix} X_0(t) \\ X_{\infty}(t) \end{bmatrix} + D w(t). \right.$$

Where $|D(A_0)| < 1$, $|D(A_{\infty})| = 1$, (A_0, B_0) is reachable

$$\text{and } \text{Im } R = \{a' X_0(0) \mid a \in \mathbb{R}^{n_0}\} \quad \ker R^* = \{a' X_{\infty}(0) \mid a \in \mathbb{R}^{n-n_0}\}$$

Moreover, $y_0(t) = C_0 X_0(t) + D w(t)$ is the p.n.d. part of y

and $y_{\infty}(t) = C_{\infty} X_{\infty}(t)$ is the p.d. — || —

$$\text{Dually, } \bar{\Sigma} : \left\{ \begin{bmatrix} \bar{X}_0(t+1) \\ \bar{X}_{\infty}(t+1) \end{bmatrix} = \begin{bmatrix} \bar{A}'_0 & 0 \\ 0 & \bar{A}'_{\infty} \end{bmatrix} \begin{bmatrix} \bar{x}_0(t) \\ \bar{x}_{\infty}(t) \end{bmatrix} + \begin{bmatrix} \bar{B}'_0 \\ 0 \end{bmatrix} \bar{w}(t) \right. \\ \left. \bar{y}(t) = [\bar{C}_0 \ \bar{C}_{\infty}] \begin{bmatrix} \bar{x}_0(t) \\ \bar{x}_{\infty}(t) \end{bmatrix} + \bar{D} \bar{w}(t) \right.$$

where (\bar{A}'_0, \bar{B}'_0) is reachable

$$\text{and } \text{Im } K = \{a' \bar{X}_0(-1) \mid a \in \mathbb{R}^{n_0}\} \quad \ker K^* = \{a' \bar{X}_{\infty}(-1) \mid a \in \mathbb{R}^{n-n_0}\}.$$

$$\text{Finally, } \bar{C}_0 = C_0 P_0 A'_0 + D B'_0 \quad \bar{C}_{\infty} = C_{\infty} P_{\infty} \bar{A}'_0$$

and $X_{\infty} = Y_{\infty}$ iff (C_{∞}, A_{∞}) is observable, or, equivalently

iff $(\bar{C}_{\infty}, \bar{A}'_0)$ is observable.

Minimality and non-minimality of finite-dimensional models.

Proposition 8.5.11: Let Σ be the linear stochastic system $\begin{cases} X(t+1) = Ax(t) + Bw(t) \\ y(t) = Cx(t) + Dw(t) \end{cases}$ and $W(z) = C(zI - A)^{-1}B + D$ its transfer fun.

Let $\Phi(z) \triangleq W(z)W(z^{-1})'$ and $X = \{ax(0) : a \in \mathbb{R}^n\} = X_0 \oplus X_\infty$ where X_∞ is the deterministic subspace of X .

Then $\Phi(e^{j\theta})$ is the spectral density of the p.i.d. part of y and $\frac{1}{2}\deg \Phi \leq \deg W \leq \dim X_0 \leq \dim X \leq \dim \Sigma$.

Moreover

(i) $\frac{1}{2}\deg \Phi = \deg W$ iff W is a minimal spectral factor

(ii) $\deg W = \dim X_0$ iff (C, A) is observable.

(iii) $\dim X_0 = \dim X$ iff $|r(A)| < 1$.

(iv) $\dim X = \dim \Sigma$ iff $x(0)$ is a basis in X

(v) $\dim X_0 = \dim \Sigma$ iff (A, B) is reachable.

In particular, if y is p.i.d., then Σ is a minimal stochastic realization of y iff (i), (ii) and (v) hold.

Otherwise, Σ is minimal iff (i), (ii) and (iv) hold.

Proposition 8.5.2: An observable system Σ with A a stability matrix is a minimal realization of y iff its steady state Kalman filter

$$\begin{cases} \hat{x}_\infty(t+1) = A\hat{x}_\infty(t) + K_\infty v(t) \\ y(t) = C\hat{x}_\infty(t) + Gv(t) \end{cases}$$

is completely reachable.

(In the sense that (A, K_∞) is reachable).