Chapter 3: Spectral representation of stationary processes.

Let $\{y(t)\}_{t\in\mathbb{Z}}$ be a stationary second-order stochastic process.

Fourier transform: $Y(\theta) = \lim_{N\to\infty} \sum_{t=-N}^{N} e^{-i\theta t} y(t)$ but the limit does not exist.
(stationarity $\Rightarrow E|y(t)|^2 = r_0 \quad \forall t$).

Solution. Integrate over interval $\Delta = (\theta_1, \theta_2)$

$$\hat{y}(\Delta) = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} Y(\theta)\,d\theta = \lim_{N\to\infty} \sum_{t=-N}^{N} \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} e^{-i\theta t} d\theta \; y(t) = \lim_{N\to\infty} \sum_{t=-N}^{N} \chi_t(\Delta)\, y(t)$$

converges in mean square

where $\chi_t(\Delta) = \begin{cases} \dfrac{e^{-i\theta_2 t} - e^{-i\theta_1 t}}{-2\pi i t} & t \neq 0 \quad \longrightarrow 0 \text{ as } t\to\infty \Rightarrow \\ \dfrac{\theta_2-\theta_1}{2\pi} & t = 0 \end{cases}$

$\hat{y}(\Delta)$ is a s.v. $= \int_\Delta d\hat{y}(\theta)$ Define: $d\hat{y}(\theta)$ - a stochastic measure on $[-\pi, \pi]$.
$\int d\hat{y}(\theta)$ - a continuous time stochastic process.

Orthogonal-increments processes

Let $x = \{x(t)\}_{t\in\mathbb{R}}$ be a scalar continuous-time process.

x has "orthogonal increments" if $E\{(x(t_2)-x(s_2))\overline{(x(t_1)-x(s_1))}\} = 0 \quad \forall s_1 < t_1 \le s_2 < t_2$

Assume also $E\{x(t)-x(s)\} = 0 \quad \forall s, t$.

Prop 3.1.1 If x has orthogonal increments, then exists a real monotone nondecreasing function F s.t. $E\{|x(t)-x(s)|^2\} = F(t)-F(s) \quad t \ge s$.
F is unique up to an additive constant.

We write this symbolically as $E\{|dx(t)|^2\} = dF(t)$.

Wiener process

If $w = \{w(t)\}_{t\in\mathbb{R}}$ has stationary orthogonal increments it is called a Wiener process

Stationarity $\Rightarrow F(t+h)-F(t) = F(h)-F(0) \quad \forall t \quad \Rightarrow F'(t) = \sigma^2 = \text{const}$.

$\Rightarrow F(t) = \sigma^2 t + \text{const}$. and $E\{|dw(t)|^2\} = \sigma^2 dt$.

so the variance grows linearly with time.

$\sigma^2 = 1 \Rightarrow$ normalized Wiener process.

Continuous time white noise = "derivative of a Wiener process"

Orthogonal stochastic measures

Let $\mathcal{R} = \{ (a,b] \mid a<b \in \mathbb{R} \}$ = the family of bounded halfopen intervals on the real line

If x is an orthogonal increments process, then it defines

an orthogonal stochastic measure on $\mathcal{R}$ by $dx((a,b]) \triangleq x(b) - x(a)$
(not positive).

and the variance measure by $m((a,b]) \triangleq F(b) - F(a)$
(positive) $= E\{|x(b)-x(a)|^2\}$

<u>Ex:</u> Normalized Wiener process $\Rightarrow$ $dw((a,b]) = w(b) - w(a)$ is a s.v.
with variance $m((a,b]) = b-a$ (i.e m = Lebesque measure)

We can also go from an orthogonal stochastic measure $d\xi$ to s.v and processes.

Define the integral $\mathcal{J}_\xi(f)$ by $\int_{\mathbb{R}} f(t)\, d\xi = \sum_{k=1}^{N} f(\Delta_k) \cdot \xi(\Delta_k)$

for simple functions $f(t) = \sum_{k=1}^{N} f(\Delta_k) I_{\Delta_k}(t)$ where $\Delta_k \in \mathcal{R}$.

Then $E\{ | \int_{\mathbb{R}} f(t)\, d\xi(t) |^2 \} = \sum_{k=1}^{N} |f(\Delta_k)|^2 \cdot m(\Delta_k) = \int_{\mathbb{R}} |f(t)|^2 dm$

This can be generalized to any $f \in L^2(\mathbb{R}, dm)$ (by taking limits and linearity)

<u>Thm 3.1.3:</u> $\mathcal{J}_\xi : L^2(\mathbb{R}; dm) \longrightarrow H(\xi)$ is a unitary map $f \mapsto \int_{\mathbb{R}} f(t)\, d\xi(t)$
$H(\xi) = \overline{\text{span}}\{\xi(\Delta) \mid \Delta \in \mathcal{R}\}$

$$E\{ \int_{\mathbb{R}} f(t)\, d\xi(t) \overline{\int_{\mathbb{R}} g(s)\, d\xi(s)} \} = \int_{\mathbb{R}} f(t) \overline{g(t)}\, dm$$

<u>Cor. 3.1.4:</u> The map assigning to any ~~For any~~ Borel set $\Delta \subset \mathbb{R}$, the random variable

$$\eta(\Delta) \triangleq \int_\Delta f(t)\, d\xi(t) = \int_\Delta I_\Delta(t) f(t)\, d\xi(t)$$

is a finite stochastic orthogonal measure iff $f \in L^2(\mathbb{R}, dm)$.

"$d\eta = f d\xi$."

Harmonic analysis of a stationary process

Let y be a scalar stationary random process (mean $= 0$). (discrete time).
and $\Lambda(\tau) = E\{y(\tau)\overline{y(0)}\}$ its covariance function

Thm 3.2.1 (Herglotz)
There is a unique finite positive measure dF on the Borel subsets of $[-\pi,\pi]$
s.t. $$\Lambda(\tau) = \int_{-\pi}^{\pi} e^{i\theta\tau}\, dF(\theta)$$

$F(\varphi) = \int_{-\pi}^{\varphi} dF(\theta)$ $(+C)$ is called the (power) spectral distribution function

In particular $\Lambda(0) = E\{|y(t)|^2\} = \int_{-\pi}^{\pi} dF(\theta) = F(\pi) < \infty$ is the variance of $y(t)$.

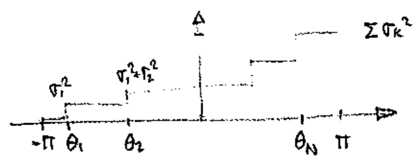
Ex 3.2.2.1 Let $y(t) = \sum_{k=-N}^{N} y_k e^{i\theta_k t}$ where $-\pi < \theta_k \leq \pi$, y_k s.v. : $E y_k \bar{y}_\ell = \delta_{k\ell}\sigma_k^2$

Then $\Lambda(\tau) = E y(t+\tau)\overline{y(t)} = \sum_{k=-N}^{N}\sum_{\ell=-N}^{N} E y_k \bar{y}_\ell\, e^{i\theta_k(t+\tau)} e^{-i\theta_\ell t} = \sum_{k=-N}^{N} \sigma_k^2 e^{i\theta_k\tau}$

so $y(t)$ is stationary.
Then $\Lambda(\tau) = \int_{-\pi}^{\pi} e^{i\theta\tau} dF(\theta)$ where $F(\theta) = \sum_{k=-N}^{N} \sigma_k^2 H(\theta-\theta_k)$ $\quad -\pi \leq \theta \leq \pi$

and $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$F(\theta)$



F monotone ⇓
Generally: $F = F_{ac} + F_s$ where $F_{ac}(\theta) = \int_{-\pi}^{\theta} \Phi(\lambda)\frac{d\lambda}{2\pi}$ is absolute continuous

$F_s(\theta) = \sum_{k=1}^{\infty} f_k H(\theta-\theta_k) + F_{ss}$ is singular.
↑ other singularities.

"summable"
If $\sum_{\tau=-\infty}^{\infty} |\Lambda(\tau)| < \infty$, then $F_s = 0$ and $\Phi(\theta) = \sum_{\tau=-\infty}^{\infty} \Lambda(\tau) e^{-i\theta\tau}$

Ex: White noise $\Lambda(\tau) = \sigma^2\delta(\tau)$ summable $\Rightarrow F_s = 0$

$dF = dF_{ac} = \sigma^2\frac{d\theta}{2\pi} \Rightarrow \Phi(\lambda) = \sigma^2$ flat density.

Spectral representation of $y(t)$

$$\mathcal{I}(e_k) = y(k)$$

Let $e_k(\theta) = e^{i\theta k}$, then $p_n = \sum_{k=-n}^{n} c_k e_k$ is a trigonometric polynomial.

Define $\mathcal{I}(p_n) = \sum_{k=-n}^{n} c_k y(k)$ and by limits define $\mathcal{I}_y : L^2([-\pi,\pi], dF) \to H(y)$
(trig. pol. dense in L^2).

<u>Thm 3.3.1</u>: There is a finite orthogonal stochastic measure $d\hat{y}$
s.t. $\mathcal{I}_{\hat{y}}(\hat{f}) = \int_{-\pi}^{\pi} \hat{f}(\theta)\, d\hat{y}(\theta)$ for $\hat{f} \in L^2([-\pi,\pi], dF)$

and in particular $y(t) = \int_{-\pi}^{\pi} e^{i\theta t}\, d\hat{y}(\theta)$ $t \in \mathbb{Z}$

$d\hat{y}$ is uniquely determined by y and satisfies

$$E\{d\hat{y}(\theta)\} = 0 \quad \text{and} \quad E\{|d\hat{y}(\theta)|^2\} = dF(\theta)$$

where F is the spectral distribution of y.

Isomorphism:
$$L^2([-\pi,\pi], dF) \xrightarrow{\mathcal{I}_{\hat{y}}} H(y)$$
$$\hat{f} \longmapsto \mathcal{I}_{\hat{y}}(\hat{f}) = \int_{-\pi}^{\pi} \hat{f}(\theta)\, d\hat{y}(\theta)$$

The shift U on $H(y)$ corresponds to multiplication with $e^{i\theta}$ in $L^2([-\pi,\pi], dF)$

$$\begin{array}{ccc} H(y) & \xrightarrow{U} & H(y) \\ \uparrow \mathcal{I}_{\hat{y}} & & \uparrow \mathcal{I}_{\hat{y}} \\ L^2([-\pi,\pi], dF) & \xrightarrow[M_e]{} & L^2([-\pi,\pi], dF) \end{array} \qquad M_e \hat{f} = e^{i\theta} \hat{f}$$

<u>That is</u>:

$$\begin{array}{ccc} \hat{f}(\theta) & \xrightarrow{M_e} & e^{i\theta}\hat{f}(\theta) \\ \downarrow & & \downarrow \\ \zeta(t) = \int_{-\pi}^{\pi} \hat{f}(\theta)\, d\hat{y}(\theta) & \xrightarrow{U} & \zeta(t+1) = \int_{-\pi}^{\pi} e^{i\theta} \hat{f}(\theta)\, d\hat{y}(\theta) \end{array}$$

In particular
$$y(t+1) = \int_{-\pi}^{\pi} \underbrace{e^{i\theta} e^{i\theta t}}_{= e^{i\theta(t+1)}} d\hat{y}(\theta)$$

Functionals of white noise

Let $\ell^2 = \{ f(t) \; ; \; t \in \mathbb{Z} \, , \; \|f\| = \left(\sum_{t=-\infty}^{\infty} |f(t)|^2 \right)^{1/2} < \infty \}$ where $\langle f,g \rangle \triangleq \sum_{t=-\infty}^{\infty} f(t) g(t)^*$

be the Hilbert space of signals with finite energy.

$w(t)$ is a white noise process if $E\{w(t)w(s)^*\} = Q\,\delta(t-s)$

and it is normalized if $Q = I$

Thm 3.5.11 Let w be a normalized white noise process. The linear functionals $\eta \in H(w)$ have the form

$$\eta = \sum_{s=-\infty}^{\infty} f(-s) w(s) \qquad f \in \ell^2 \text{ is unique,}$$

and $\mathcal{J}_w : \ell^2 \to H(w)$, $f \to \eta = \mathcal{J}_w f$, is unitary.

Let $T : \ell^2 \longrightarrow \ell^2$ be the translation operator in ℓ^2.

$f(t) \longmapsto Tf(t) = f(t+1)$

Then the diagram

$$\begin{tikzcd} \ell^2 \arrow[r, "T"] \arrow[d, "\mathcal{J}_w"'] & \ell^2 \arrow[d, "\mathcal{J}_w"] \\ H(w) \arrow[r, "U"] & H(w) \end{tikzcd}$$

commutes, so

$$\eta(t) \triangleq U^t \eta = \sum_{s=-\infty}^{\infty} f(t-s) w(s) = \mathcal{J}_w(T^t f)$$

Proofs $\{w(s)\}_{s \in \mathbb{Z}}$ form an ON-basis for $H(w)$ $\Rightarrow$ $\eta = \sum_{s=-\infty}^{\infty} \eta_s w(s)$

where $\eta_s = \langle \eta, w(s) \rangle = f(-s)$.

Then since $U^{-t} w(s) = w(s-t)$

$$E\{\eta(t)\overline{w(s)}\} = \langle U^t \eta, w(s) \rangle = \langle \eta, U^{-t} w(s) \rangle = \langle \eta, w(s-t) \rangle = f(-(s-t)) = f(t-s).$$

$\square$

Then the representation of $\eta(t)$ in terms of $f(t) \in \ell^2$

is related to the representation of $\eta(t)$ in terms of $\hat{f}(\theta) \in L^2([-\pi,\pi], \frac{d\theta}{2\pi})$

by the Fourier transform...

w normalized white noise

$\Rightarrow dF(\theta) = \frac{d\theta}{2\pi}$

The Fourier transform

$e_t(\theta) \doteq e^{i\theta t}$ $\quad t \in \mathbb{Z}$ form a complete ON-basis in $L^2([-\pi,\pi), \frac{d\theta}{2\pi})$.

Thm 3.5.3: (Fourier-Plancherel)

The Fourier transform $\mathcal{F}: \ell^2 \to L^2([-\pi,\pi], \frac{d\theta}{2\pi})$ is a unitary map.

$$f \mapsto \hat{f} = \mathcal{F}(f) = \sum_{t=-\infty}^{\infty} e^{-i\theta t} f(t).$$

(the sum being convergent ($\forall f \in \ell^2$) in the metric of the space $L^2([-\pi,\pi], \frac{d\theta}{2\pi})$

is a norm preserving and onto, i.e. a unitary, map from ℓ^2 onto $L^2([-\pi,\pi], \frac{d\theta}{2\pi})$)

Parsevals identity — $\displaystyle\sum_{t=-\infty}^{\infty} |f(t)|^2 = \int_{-\pi}^{\pi} |\hat{f}(\theta)|^2 \frac{d\theta}{2\pi}$

Let T - translation operator $\quad T(f)(t) \doteq f(t+1)$

Thm 3.5.5. Let w be a normalized white noise process.

The unitary map $\mathcal{J}_w: \ell^2 \to H(w)$

$$f \mapsto \eta = \sum_{s=-\infty}^{\infty} f(-s)\, w(s)$$

admits a factorization as the composite map $\mathcal{J}_w = \mathcal{J}_{\hat{w}} \mathcal{F}$

The following diagram commutes

$$\begin{tikzcd}
H(w) \arrow[r, "U"] & H(w) \\
L^2([-\pi,\pi], \frac{d\theta}{2\pi}) \arrow[u, "\mathcal{J}_{\hat{w}}"] \arrow[r, "M_e"] & L^2([-\pi,\pi], \frac{d\theta}{2\pi}) \arrow[u, "\mathcal{J}_{\hat{w}}"'] \\
\ell^2 \arrow[u, "\mathcal{F}"] \arrow[r, "T"] \arrow[uu, bend left=60, "\mathcal{J}_w"] & \ell^2 \arrow[u, "\mathcal{F}"'] \arrow[uu, bend right=60, "\mathcal{J}_w"']
\end{tikzcd}$$