Wiener filters

Chapter 4

Assume: x and y are jointly stationary second order processes.

The joint spectral distribution function F is absolutely continuous with known spectral density matrix
$$\Phi(e^{i\theta}) = \begin{bmatrix} \Phi_x & \Phi_{xy} \\ \Phi_{yx} & \Phi_y \end{bmatrix}(e^{i\theta})$$

Two estimation problems.

1. Acausal: $\hat{x}(t) \triangleq E[x(t) \mid H(y)]$ based on all values of y.
2. Causal: $\hat{x}_-(t) \triangleq E[x(t) \mid H_t^-(y)]$ based on all past values of y.

Acausal linear estimation

Lemma 4.1.1: Assume that $y = w$ is normalized white noise.
Then the best linear estimator of $x(t)$ given $H(w)$ is given by
$$\hat{x}(t) = E^{H(w)} x(t) = \sum_{s=-\infty}^{\infty} F(t-s) w(s)$$
where $F(t) = \Lambda_{xw}(t) = \int_{-\pi}^{\pi} e^{i\theta t} \Phi_{xw}(\theta) \frac{d\theta}{2\pi} \qquad \forall t \in \mathbb{Z}.$

Proof: Thm 3.5.1 $\Rightarrow$ $\hat{x}(t) = \sum_{s=-\infty}^{\infty} F(t-s) w(s) \quad \forall t$

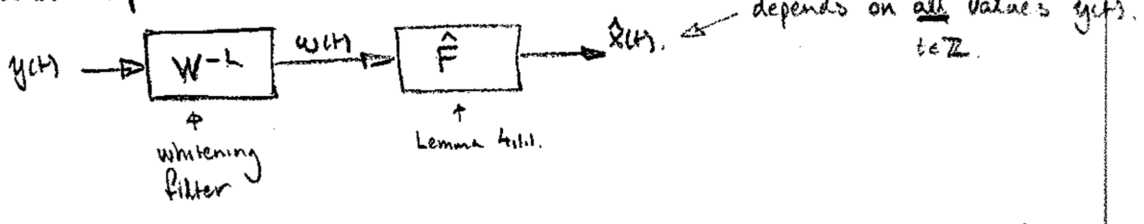
Orthogonal projection lemma $\Rightarrow$ $E(x(t) - \hat{x}(t)) w(\tau)' = \Lambda_{xw}(t-\tau) - \underbrace{\sum F(t-s)\delta(s-\tau)}_{= F(t-\tau)} = 0$. $\square$

So the problem is easy if y is white noise, or if y can be transformed into white noise.

Def: y is orthonormalizable if $\exists$ norm. white noise, jointly stationary with y, such that $H(y) = H(w)$.

If y is orthonormalizable, then $y(t) = \sum_{s=-\infty}^{\infty} \check{W}(t-s) w(s)$ — shaping filter representation

Cascade implementation of Wiener filter



Thm 4.2.1: (Spectral factorization) A stationary m-dim. process y is orthonormalizable iff.

1. y's spectral distribution function is absolutely continuous and the spectral density matrix Φ has constant rank p a.e. on $[-\pi, \pi]$. see Section 3.7.
2. there exists $m \times p$-functions W s.t. $\Phi(e^{i\theta}) = W(e^{i\theta}) W(e^{i\theta})^*$ a.e. on $[-\pi, \pi]$

Spectral factor — In general there are many!

If W has a.e. linearly independent columns it is called a full rank spectral factors.

Easy to show that $w(t)$ is white noise:

Let $\hat{w}(\Delta) \triangleq \int_\Delta W^{-1}(e^{i\theta})d\hat{y}(e^{i\theta})$ for an arbitrary Borel set $\Delta \subset [-\pi,\pi]$

$$\Rightarrow E\{\hat{w}(\Delta_1)\hat{w}(\Delta_2)^*\} = \int_{\Delta_1\cap\Delta_2} W^{-1} \underbrace{E|d\hat{y}|^2}_{=dF=\Phi\frac{d\theta}{2\pi}}(W^{-1})^* = \int_{\Delta_1\cap\Delta_2} I\cdot\frac{d\theta}{2\pi} = I\cdot\frac{|\Delta_1\cap\Delta_2|}{2\pi}$$

Lebesgue measure

Causal Wiener filtering

Do whitening as before: But since $\hat{x}_-(t) = E^{H_t^-(y)}x(t) \in H_t^-(y)$

future values for $y(t+1), y(t+2), \dots$ does not affect $\hat{x}_-(t)$ and we must have that

$$\hat{x}_-(t) = \sum_{s=-\infty}^{\infty} F(t-s)w(s) \quad \text{for a function } \underbrace{F \text{ s.t. } F(\tau)=0 \ \forall \tau<0}_{\text{Causal functions}}.$$

and the white noise should be

causally equivalent to y: $H_t^-(w) = H_t^-(y) \quad \forall t\in\mathbb{Z}$.

Lemma 4.1.4: Assume that $y=w$ is normalized white noise.

Then the best linear estimator of $x(t)$ given w up to time $t-1$

$$\hat{x}(t) = E^{H_t^-(w)}x(t) = \sum_{s=-\infty}^{\infty} F(t-s)w(s)$$

where
$$F(t) = \begin{cases} \Lambda_{xw}(t) & t>0 \\ 0 & t\leq 0. \end{cases} \quad \text{(strictly causal)}$$

From the acausal case we know.

$\boxed{\Phi \text{ factorizable necessary cond}}$

When can we find a causally equivalent white noise w?

- We need some special spectral factors.

Let $\ell_p^{2+} \triangleq \{f(t)\in\ell_p^2 : f(t)=0 \ \forall t<0\}$ – the causal functions in ℓ_p^2.

We know $\hat{f}(\theta) = \mathcal{F}(f) = \sum_{t=-\infty}^{\infty} f(t)e^{-i\theta t}$, similarly let $F(z) = \sum_{t=-\infty}^{\infty} f(t)z^{-t}$.

Converges on the unit circle, but otherwise?!

Def: The p-dim. Hardy space is given by

$$H_p^2(\mathbb{D}) \triangleq \{F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}, |z|>1. \ \sum_{k=0}^{\infty}|f(k)|^2<\infty\}$$

analytic functions

and

$$\bar{H}_p^2(\mathbb{D}) \triangleq \{F \mid F(z^{-1})\in H_p^2(\mathbb{D})\}.$$

Introduce the norm $\|F\| = \lim_{\rho\searrow 1}\|F_\rho\|_{L^2}$ where $F_\rho: \theta \mapsto F(\rho e^{i\theta})$ for $\rho>1$

and
$$\|F_\rho\|_{L^2}^2 = \int_{-\pi}^{\pi}|F(\rho e^{i\theta})|^2\frac{d\theta}{2\pi} = \int_{-\pi}^{\pi}\Big|\sum_{k=0}^{\infty} f(k)\rho^{-k}e^{-ik\theta}\Big|^2\frac{d\theta}{2\pi} = \sum_{k=0}^{\infty}|f(k)|^2\rho^{-2k}$$

Then there is a unitary map between ℓ_p^{2+} and $H_p^2(\mathbb{D})$.

$(\ell_p^{2-}$ and $\bar{H}_p^2(\mathbb{D}))$

$\Phi(\theta) = \sum_{\tau=-\infty}^{\infty} \Lambda(\tau) e^{-i\theta\tau}$ $\xrightarrow[\text{notation}]{\text{change of}}$ $\Phi(e^{i\theta}) = \sum_{\tau=-\infty}^{\infty} \Lambda(\tau) e^{-i\theta\tau}$.

$\Phi(z) = \sum_{\tau=-\infty}^{\infty} \Lambda(\tau) z^{-t}$ $\Lambda(\tau)$ real $\Rightarrow$ $\Phi(1/z)' = \Phi(z)$ (since $\Lambda(-\tau) = \Delta(\tau)'$)

Parahermitian symmetry.

Def 4.1.2. A process y admitting a causally equivalent white noise process is called a forward purely nondeterministic process. (p.n.d)

Thm 4.4.1. Given an m-dim. process y, there is a normalized r-dim white noise w s.t. $H_t^-(y) \subset H_t^-(w)$ $t \in \mathbb{Z}$ only if there are $m \times r$ analytic spectral factors of Φ_y

i.e W s.t. $\phi(z) = W(z)W(z^{-1})'$ where the rows of $W \in H_r^2$.

Conversely, if ϕ admits analytic spectral factors, the process is p.n.d. and in particular there exists a normalized white process w_- such that $d\hat{y} = W_- d\hat{w}_-$, where W_- is analytic and of full rank p a.e and $H_t^-(y) = H_t^-(w_-)$.

proof. Hemtal.

The construction of w_- follows from the Wold decomposition

Def 4.5.1: A second-order process y is purely deterministic (p.d) if the one-step prediction error $e(t) \triangleq y(t) - E^{H_t^-(y)} y(t) = 0$ $\forall t$ a.s.

Then if y is p.d then $H_{t+1}^-(y) = H_t^-(y)$ and stationarity $\Rightarrow$ $H_t^-(y) = H(y)$ $\forall t \in \mathbb{Z}$

Define: Remote past $H_{-\infty}(y) \triangleq \bigcap_{t \leq k} H_t^-(y) = \lim_{t \to -\infty} H_t^-(y)$

Remote future $H_{+\infty}(y) \triangleq \bigcap_{t \geq k} H_t^+(y) = \lim_{t \to \infty} H_t^+(y)$.

Then y is p.d iff $H_{-\infty}(y) = H(y)$ and $H_{-\infty}(y)$ is invariant under both U and U^*

Similarly for backward p.d.

$\bar{e}(t) \triangleq y(t) - E^{H_{t+1}^+(y)} y(t)$ is the backward prediction error.

Then y is backward p.d. iff. $H_{+\infty}(y) = H(y)$.

If $H_{-\infty}(y) = H_{+\infty}(y)$ the process y is called reversible.

Prop 4.5.11: Every full rank p.n.d process is reversible.

Proof idea: $\Lambda(-\tau) = \Lambda(\tau)'$ $\Rightarrow$ reversing time $\Leftrightarrow$ $\Phi \mapsto \Phi'$

p.n.d + full rank $\Leftrightarrow$ $\int_{-\pi}^{\pi} \log\det\Phi \, d\theta > -\infty$.

Theorem 4.5.4: (Wold) (simplified) Assume y is a stationary second-order process.

$H_{-\infty}(y) = \{0\}$ iff $H_t^-(y) \subset H_t^-(w)$ $\forall t \in \mathbb{Z}$ for some white noise w and there is a unique (modulo mult. by a constant orthogonal matrix) normalized white noise w_- s.t. $H_t^-(y) = H_t^-(w_-)$.

Proof: ($\Rightarrow$) Define $W_t = H_{t+1}^-(y) \ominus H_t^-(y)$ $\forall t \in \mathbb{Z}$ (non-trivial since $H_\infty(y) = 0$).

Then $H_{t+1}^-(y) = W_t \oplus W_{t-1} \oplus \dots \oplus W_s \oplus H_s^-(y)$ $s<t$. $W_t \perp W_s$ $s \neq t$

With $W \triangleq H_1^-(y) \ominus H_0^-(y)$, then $W_t = U^t W$ $t \in \mathbb{Z}$

then W is a wandering subspace for the shift U.

$\eta(t) \in H_{t+1}^-(y) \Rightarrow \eta(t) = \hat{\eta}_s(t) + \tilde{\eta}_s(t)$ where $\hat{\eta}_s(t) \in \bigoplus_{k=s}^{t} W_s$, $\tilde{\eta}_s(t) \in H_s^-(y)$

$H_s^-(y) \to H_{-\infty}(y) = \{0\}$ as $s \to -\infty$ $\Rightarrow \tilde{\eta}_s(t) \to 0$.

$\Rightarrow \eta(t) = \lim_{s \to -\infty} \hat{\eta}_s(t) = \sum_{s=-\infty}^{t} E^{W_s} \eta(t)$ i.e. $H_{t+1}^-(y) = \bigoplus_{s=-\infty}^{t} W_s$

$\|\hat{\eta}_s(t)\| \leq \|\eta(t)\| \Rightarrow \hat{\eta}_s(t)$ cauchy-sequence $\Rightarrow$ konvergence

Any orthonormal basis $\{w_1 \dots w_p\}$ of W shifted in time is a normalized white noise w_-

($\Leftarrow$) Lemma 4.5.5: The remote past of a white noise process is trivial. □

What if y is not p.n.d.?

Corollary 4.5.9 (Wold decomposition)

Every stationary vector process y admits a decomposition $y(t) = v(t) + z(t)$ $t \in \mathbb{Z}$ where v and z are uncorrelated, v is forward p.d and z is forward p.n.d.
($H(v) = H_{-\infty}(y)$) ($H_{-\infty}(z) = 0$)

Symmetric result hold for backward p.d. and p.n.d decomposition.

The white noise w_- corresponds to a special spectral factor W_-. What characterize W_-?

If $H_0^-(w) = H_0^-(y)$, $d\hat{y} = W d\hat{w}$ $\Rightarrow \underbrace{\overline{\text{span}}\{e^{i\theta t} e_k \mid {k=1..p \atop t \leq 0}\}}_{= H_p^2} = \overline{\text{span}}\{e^{i\theta t} W_k \mid {k=1\dots p \atop t \leq 0}\}$

Note $F_0 = F(\infty) \neq 0$

Def 4.6.1.1 An $m \times p$ F with rows $F_k \in H_p^2$ is called outer if $\overline{\text{span}}\{z^t F_k \mid {k=1\dots m \atop t \leq 0}\} = H_p^2$

A $p \times p$ $Q \in H^\infty_{p\times p}$ with unitary values on the unit circle, $Q(e^{i\theta}) Q(e^{i\theta})^* = I$ is called inner.

Here $H^\infty_{m\times p} = \{F$ uniformly bounded in $\{|z|>1\}$ $\mid F_k \in H_p^2, k=1..m\}$.

Theorem 4.6.5: (Inner-outer factorization)

Every matrix function $F \in H^2_{m\times p}$ of **full column rank** a.e. has a factorization $F = F_- Q$ where F_- is outer and Q is inner. F_- and Q are unique up to $p \times p$ constant orthogonal factors.

Compare: Polar form factorization.

Theorem 4.6.8: Assume $\Phi(z)$ is an $m \times m$ spectral density matrix of rank p a.e. admitting analytic spectral factors. Then $\Phi(z)$ admits an outer spectral factor W_- of dimension $m \times p$. This is the unique outer factor of $\Phi(z)$, modulo right multiplication by a constant $p \times p$ unitary matrix.
Every full-rank analytic spectral factor W can be written $W = W_- Q$ where Q is a function uniquely determined by W mod G.
All other analytic spectral factors of dim. $m \times r$ $(r \geq p)$ are of the form $W = W_- R$ where R is an $p \times r$ unilateral inner function.

Def 4.6.6: A function $R \in H^\infty_{p\times r}$ where $r \geq p$ s.t. $R(e^{i\theta}) R(e^{i\theta})^* = I_p$ is called a unilateral inner function

What does an outer function look like?

Def 4.6.10: Let $F \in H^2_{m\times p}$ have full column rank a.e. An $\alpha \in \mathbb{C}$ s.t. $|\alpha| > 1$ is a (right) zero of F, if $\exists\, v \in \mathbb{C}^p$ $(v \neq 0)$ s.t. $F(\alpha) v = 0$.

Thm 4.6.11: An outer function have no zeros in $\{|z|>1\}$ (including infinity).
A rational $f \in H^2$ is outer iff it has no poles in $\{|z| \geq 1\}$ and no zeros in $\{|z|>1\}$ (including infinity).

What does an inner function look like?

A real scalar inner function is of the form $Q(z) = c\, B(z)\, S(z)$

where $c = \pm 1$ $\quad B(z) = \prod_{k=1}^{\infty} \frac{1-\alpha_k z}{z - \bar{\alpha}_k}$ $\quad |\alpha_k| < 1$ Blaschke product

and $S(z) = \exp\left\{ - \int_{-\pi}^{\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}}\, d\mu(e^{i\theta}) \right\}$ where μ is a finite positive measure whose support has Lebesgue measure zero.

The determinant of a matrix-valued inner function is a scalar-valued inner function.

Theorem 4.6.4: (Beurling-Lax)

Every full-range p.n.d z^{-1}-invariant subspace $\mathcal{Y} \subset L_p^2$ has the form

$$\mathcal{Y} = \{ fQ \mid f \in H_p^2 \} \triangleq H_p^2 Q$$

where Q is a $p \times p$-matrix function with unitary values on the unit circle, i.e. $Q(e^{i\theta}) Q(e^{i\theta})^* = I$

If $\mathcal{Y} \subset H_p^2$, then Q is actually inner.

In any case, Q is uniquely determined by $\mathcal{Y}$ modulo a constant unitary factor.

Def: $\mathcal{Y} \subset L_p^2$ is full range if $\bigvee_{t \in \mathbb{Z}} z^t \mathcal{Y} = L_p^2$.

Equivalently $\mathcal{Y}$ is full range if $\mathcal{Y}^\perp$ is p.n.d, i.e. $\bigcap_{t \in \mathbb{Z}} z^{-t} \mathcal{Y}^\perp = 0$.

Proof: The proof is based on the general version of Thm 4.5.4 (Wold)

Regard L_p^2 as the frequency domain representation of the Hilbert space $H(w)$ generated by some p-dimensional white process w.

i.e. $H(w) = \mathcal{I}_{\hat{w}}(L_p^2)$

Define $Y \triangleq \mathcal{I}_{\hat{w}}(\mathcal{Y}) \Rightarrow Y \subset H(w)$

Wold thm $\Rightarrow$ $Y = H^-(u)$ $\quad$ u white noise.

$\mathcal{Y}$ full range $\Rightarrow$ $H(u) = H(w)$. $\quad$ $\dim(u) = \dim(w)$

Lemma 4.2.4 $\Rightarrow$ $d\hat{u} = Q\, d\hat{w}$

$\Rightarrow \Phi_w = QQ^* = I$ since w white noise.

$$\hat{f} \in H_p^2 \Rightarrow \mathcal{I}_{\hat{u}}(\hat{f}) = \int_{-\pi}^{\pi} \hat{f}(\theta) \underbrace{d\hat{u}(\theta)}_{= Q\, d\hat{w}(\theta)} = \mathcal{I}_{\hat{w}}(\hat{f} Q).$$

$$\Rightarrow H^-(u) = \mathcal{I}_{\hat{u}}(H_p^2) = \mathcal{I}_{\hat{w}}(H_p^2 Q) \quad \Rightarrow \boxed{\mathcal{Y} = H_p^2 Q}$$

$\mathcal{Y} \subset H_p^2$ iff $H^-(u) \subset H^-(w)$

Thm 4.4.1 $\Rightarrow$ Q is analytic.