Chapter 6 Linear stochastic systems.

Deterministic realization theory.

Let $\Sigma \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$

If $x(0)=0$ and $u(t) = \begin{cases} u(0) & t=0 \\ 0 & t>0 \end{cases}$ then $y(t) = R_t u(0)$ $t \geq 0$.

where $R_0 = D$, $R_k = CA^{k-1}B$ $k=1,2\ldots$ is the impulse response of Σ.

Transfer function $R(z) = \sum_{k=0}^{\infty} R_k z^{-k} \longrightarrow C(zI-A)^{-1}B+D$ around infinity.

Realization: Determine Σ from R.

Let the McMillan degree $\delta(R)$ be the dimension of a minimal realization

Now
$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} R_1 & R_2 & R_3 & \cdots \\ R_2 & R_3 & R_4 & \cdots \\ R_3 & R_4 & R_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\triangleq \mathcal{H}} \begin{bmatrix} u(-1) \\ u(-2) \\ \vdots \\ u(-N) \\ 0 \\ \vdots \end{bmatrix}$$

$u \longrightarrow \boxed{R(z)} \longrightarrow y.$

$D = R_0$, (A,B,C) remain to be determined

$\mathcal{H}$ is a Hankel matrix, and if generated by some Σ then

$$\mathcal{H} = \begin{bmatrix} CB & CAB & CA^2B & \cdots \\ CAB & CA^2B & & \\ CA^2B & & & \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}}_{\text{observability matrix } \mathcal{O}} \underbrace{[B \;\; AB \;\; A^2B \;\; \cdots]}_{\text{reachability matrix } \mathcal{R}} \triangleq \mathcal{O}\mathcal{R}$$

Abstractly

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\mathcal{H}"] \arrow[dr, "\mathcal{R}"'] & & \mathcal{Y} \\ & X \arrow[ur, "\mathcal{O}"'] & \end{tikzcd}$$

$\mathcal{Y} = \{ y(t) \mid y(t) = 0 \quad t<0 \}$

$\mathcal{U} = \{ u(t) \mid u(t) = 0 \quad t \geq 0,\ t < -N \}$

$\mathcal{H} = E^{H^+(y)}|_{H^-(u)}$ Hankel operator.

$$\mathcal{R} \begin{bmatrix} u(-1) \\ u(-2) \\ \vdots \end{bmatrix} = Bu(-1) + ABu(-2) + \ldots = x(0)$$

Σ is completely reachable if $\mathcal{R}$ is surjective (onto), i.e. $\operatorname{Im} \mathcal{R} = X$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} x(0) = \mathcal{O} x(0)$$

Σ is completely observable if $\mathcal{O}$ is injective (one-to-one), i.e. $\operatorname{Ker} \mathcal{O} = 0$.

Theorem 6.1.5: A realization Σ of R is minimal iff it is both reachable & observable

Cor. 6.1.4: The McMillan degree $\delta(R)$ is equal to rank$(\mathcal{H})$.

Stochastic state-space realizations

Now $\quad \Sigma \begin{cases} x(t+1) = Ax(t) + Bw(t) \\ y(t) = Cx(t) + Dw(t) \end{cases} \quad \{x(t)\}_{t\in\mathbb{Z}}$ n-dim state process, $\{y(t)\}_{t\in\mathbb{Z}}$ m-dim. stationary process.

where A is a <u>stability matrix</u> - all eigenvalues of A in the open unit disc.

$w(t)$ is a p-dim. normalized white noise - $E\{w(t)\}=0 \quad E\{w(t)w(s)'\} = I\delta_{ts}$.

This is a causal representation of y

A stability matrix $\Longrightarrow \begin{cases} x(t) = \sum_{j=-\infty}^{t-1} A^{t-1-j}Bw(j) \\ y(t) = \sum_{j=-\infty}^{t-1} CA^{t-1-j}Bw(j) + Dw(t) \end{cases}$

So if $H(w) = \overline{\text{span}}\{w_i(t) \mid t\in\mathbb{Z}, i=1..p\}$ then $H(x)\subset H(w)$, $H(y)\subset H(w)$

Furthermore if $H_t^-(w) = \overline{\text{span}}\{w_i(s) \mid s<t, i=1..p\}$ then $H_{t+1}^-(x) \subset H_t^-(w)$, $H_t^-(y)\subset H_t^-(w)$

$\Rightarrow$ <u>Causality</u>: $H_t^+(w) = \overline{\text{span}}\{w_i(s) \mid s\geq t, i=1..p\} \perp \underbrace{(H_{t+1}^-(x) \vee H_t^-(y))}_{\text{called } S_t \text{ later.}} \quad \forall t\in\mathbb{Z}$

Σ evolves forward in time

<u>State space</u>: $X_t = \text{span}\{x_1(t), \dots, x_n(t)\} \subset H(w) \quad t\in\mathbb{Z}$

$\dim X_t = n$ iff $E\{|\sum a_k x_k(t)|^2\} = 0 \Rightarrow a_1 \dots a_n = 0$.

i.e. iff $\underbrace{P = E\{x(t)x(t)'\}}_{\text{State covariance}} > 0$ where P solves $P = APA' + BB'$.

A stab. matrix $\Rightarrow P = \sum_{j=0}^{\infty} A^{j-1}BB'(A')^{j-1} = \mathcal{R}\mathcal{R}' \quad \Rightarrow \boxed{P>0 \Leftrightarrow (A,B) \text{ reachable}}$

<u>Prop. 6.2.3</u>: The spaces $S_t \triangleq H_{t+1}^-(x)\vee H_t^-(y)$ and $\bar{S}_t \triangleq H_t^+(x)\vee H_t^+(y)$ are conditionally orthogonal given X_t, i.e. $S_t \perp \bar{S}_t \mid X_t \quad t\in\mathbb{Z}$.

<u>proof</u>: $(s\geq t)$ $x(s) = A^{s-t}x(t) + \sum_{j=t}^{s-1} A^{s-1-j}Bw(j)$, $y(s) = CA^{s-t}x(t) + \sum_{j=t}^{s-1} CA^{s-1-j}Bw(j) + Dw(s)$

$\Rightarrow E^{S_t} b'\begin{bmatrix} x(s) \\ y(s)\end{bmatrix} = b'\begin{bmatrix} A^{s-t} \\ CA^{s-t}\end{bmatrix} x(t) = E^{X_t} b'\begin{bmatrix} x(s) \\ y(s)\end{bmatrix} \quad \forall b\in\mathbb{R}^{n+m}$

$\Rightarrow E^{S_t}\lambda = E^{X_t}\lambda \quad \forall t\in\mathbb{Z}$ and $\lambda\in\bar{S}_t$

$X_t \subset H_{t+1}^-(x) \subset S_t \quad \Rightarrow \quad E^{S_t\vee X_t}\lambda = E^{X_t}\lambda$

□.

Anti-causal state-space models:

Note: $S_t \perp \bar{S}_t \mid X_t$ is symmetric w.r.t. time.

$$S_t \perp \bar{S}_t \mid X_t \quad \Rightarrow \quad E^{\bar{S}_t}\lambda = E^{X_t}\lambda \quad \forall \lambda \in S_t,\ t \in \mathbb{Z}$$

$$S_t = H^-_{t+1}(x) \vee H^-_t(y) = H^-_t(z) \quad \text{where } z(t) \triangleq \begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix}$$

Now $z(t) = \begin{bmatrix} A \\ C \end{bmatrix} x(t) + \begin{bmatrix} B \\ D \end{bmatrix} w(t) = E^{H^-_{t-1}(z)} z(t) + \nu(t) =$ one-step predictor + innovation process.

Symmetrically,

$$\bar{S}_t = H^+_t(x) \vee H^+_t(y) = H^+_t(\bar{z}) \quad \text{where } \bar{z}(t) \triangleq \begin{bmatrix} \bar{x}(t-1) \\ y(t) \end{bmatrix} \quad \text{and } \bar{x}(t) \triangleq P^{-1}x(t+1)$$

Now $\bar{z}(t) = \hat{\bar{z}}(t) + \bar{\nu}(t) =$ backward one-step predictor + backward innovation process.

$$b'\bar{z}(t) \in S_{t+1} \quad \Rightarrow \quad b'\hat{\bar{z}}(t) = E^{H^+_{t+1}(\bar{z})} b'\bar{z}(t) = E^{\bar{S}_{t+1}} b'\bar{z}(t) \underbrace{=}_{E^{\bar{S}_t}\lambda = E^{X_t}\lambda \ \forall \lambda \in S_t} E^{X_{t+1}} b'\bar{z}(t)$$

$$= b' E\{\bar{z}(t)x(t+1)'\} E\{x(t+1)x(t+1)'\}^{-1} x(t+1)$$

$E\,x(t)x(t+1)' = PA'$

$E\,y(t)x(t+1)' = CPA' + DB'$

$$= b' \begin{bmatrix} A' \\ CPA' + DB' \end{bmatrix} P^{-1} x(t+1)$$

$$\Rightarrow \hat{\bar{z}}(t) = \begin{bmatrix} A' \\ \bar{C}' \end{bmatrix} \bar{x}(t) \quad \text{where } \bar{C} \triangleq CPA' + DB'$$

Theorem 6.3.1: Consider the forward state-space model with state covariance $P = E\{x(t)x(t)'\}$. Then $\bar{x}(t) = P^{-1}x(t+1)$ is the state process of the backward system

$$\bar{\Sigma} \quad \begin{cases} \bar{x}(t-1) = A'\bar{x}(t) + \bar{B}\bar{w}(t) \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{w}(t) \end{cases}$$

$\bar{P} = E\{P^{-1}x(t+1)x(t+1)'P^{-1}\} = P^{-1}PP^{-1} = P^{-1}$

with state covariance $\bar{P} \triangleq E\{\bar{x}(t)\bar{x}(t)'\} = P^{-1}$

$\bar{B}, \bar{D}$ are determined by a minimum-rank factorization

$$\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}' = \begin{bmatrix} \bar{P} - A'\bar{P}A & C' - A'\bar{P}\bar{C}' \\ C - \bar{C}\bar{P}A & \Lambda_0 - \bar{C}\bar{P}\bar{C}' \end{bmatrix} \qquad \Lambda_0 \triangleq E\,y(t)y(t)'$$

and $\bar{w}$ is a centered normalized white noise.

$\bar{\Sigma}$ is a backward model in the sense that $H^-_t(\bar{w}) \perp \underbrace{H^+_t(x) \vee H^+_t(y)}_{\bar{S}_t} \quad t \in \mathbb{Z}$

Proof. $\bar{\nu}(t) \triangleq \bar{z}(t) - \hat{\bar{z}}(t)$ is white noise: $E\,\bar{\nu}(t)\bar{\nu}(s)' = \bar{V}\delta_{t,s}$.

$$\bar{V} = E(\bar{z}(t) - \hat{\bar{z}}(t))(\bar{z}(t) - \hat{\bar{z}}(t))' = E\,\bar{z}(t)\bar{z}(t) - E\,\hat{\bar{z}}(t)\hat{\bar{z}}(t)' = E\left\{\begin{bmatrix} \bar{x}(t-1) \\ y(t) \end{bmatrix}\begin{bmatrix} \bar{x}(t-1) \\ y(t) \end{bmatrix}'\right\} - \begin{bmatrix} A' \\ \bar{C} \end{bmatrix} \bar{P} [A\ \bar{C}']$$

$$= \begin{bmatrix} \bar{P} - A'\bar{P}A & C' - A'\bar{P}\bar{C}' \\ C - \bar{C}\bar{P}A & \Lambda_0 - \bar{C}\bar{P}\bar{C}' \end{bmatrix} = E\left\{\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix} \bar{w}(t)\bar{w}(t)' [\bar{B}'\ \bar{D}']\right\} \quad \text{for } \bar{w} \text{ a normalized white noise}$$

$\bar{\Sigma}$ has the antistable transfer function $\bar{W}(z) = \bar{C}(z^{-1}I - A')^{-1}\bar{B} + \bar{D}$.

$(w, \bar{w})$ are called the forward and backward generating processes corresp. to $\{X_t\}_{t\in\mathbb{Z}}$

Theorem 6.4.1: Let $(w, \bar{w})$ be the pair generating Σ and $\bar{\Sigma}$.

Then $V \triangleq E\{\bar{w}(t)w(t)'\}$

satisfies $VV' = I - \bar{B}'P\bar{B}$ and $V'V = I - B'\bar{P}B$

Moreover $\bar{w}(t) = \bar{B}'x(t) + Vw(t)$, $w(t) = B'\bar{x}(t) + V'\bar{w}(t)$.

Finally $H(\bar{w}) = H(w)$.

Now $\xrightarrow{w} \boxed{K(z)} \xrightarrow{\bar{w}}$ with $K(z) = \bar{B}'(zI - A)^{-1}B + V$ the structural function

all-pass filter $\quad KK^* = I \Rightarrow K^* = K^{-1}$

$\xrightarrow{\bar{w}} \boxed{K^{-1}(z)} \xrightarrow{w}$ $\quad K(z)^{-1} = B'(z^{-1}I - A')^{-1}\bar{B} + V'$

Theorem 6.4.3: $W = \bar{W}K$

Since K rational $\Rightarrow K(z) = \bar{M}(z)M(z)^{-1}$ where $M, \bar{M}$ are matrix polynomials

All roots of $\det M(z)$ are in $|z| < 1$

———— " — $\det \bar{M}(z)$ are in $|z| > 1$.

Cor 6.4.4: $\exists$ matrix polynomial $N(z)$ s.t.

$$W(z) = N(z)M(z)^{-1}$$
$$\bar{W}(z) = N(z)\bar{M}(z)^{-1}$$

Same zeros
Reflected poles.