Coordinate free representations

Forward & backward systems have the same state space families

Note $X_t = \{a' x(t) \mid a \in \mathbb{R}^n\} = \{a' \bar{x}(t-1) \mid a \in \mathbb{R}^n\}$

Let $\mathbb{H} \triangleq H(w)$ be the Ambient space with the shift $\mathcal{U}$ s.t. $\mathcal{U} w_i(t) = w_i(t+1)$.

Consider $t=0$: Stationarity $\Rightarrow$ All properties depend on relative time.

With $X = X_0 \Rightarrow X_t = \mathcal{U}^t X$.

Let $H \triangleq H(y)$, $H^- \triangleq H_0^-(y)$, $H^+ \triangleq H_0^+(y)$.

Clearly $H = H^- \vee H^+ \subset \mathbb{H}$, $U^{-1} H^- \subset H^-$, $U H^+ \subset H^+$.

A $\xi \in X$ is unobservable if it cannot be distinguished from zero by observing the future of y, i.e. if $\xi \perp H^+$

Def 6.6.1: A state space X of a linear stoch. system is observable if $X \cap (H^+)^\perp = 0$.

A $\xi \in X$ is unconstructible if it cannot be distinguished from zero by observing the past of y, i.e. if $\xi \perp H^-$

Def. 6.6.1: A state space X of a linear stoch. system is constructible if $X \cap (H^-)^\perp = 0$.

Thm 6.6.2: Let X be the state space of some pair $(\Sigma, \bar{\Sigma})$.

Then X is observable iff $\bigcap_{t=0}^{\infty} \ker CA^t = 0$

and X is constructible iff $\bigcap_{t=0}^{\infty} \ker \bar{C}(A')^t = 0$.

Proof: $\xi \in X \cap (H^+)^\perp \Rightarrow \xi = a' x(0) \perp b' y(t) \quad \forall b \in \mathbb{R}^m,\ t = 0, 1, \dots$

$E\{y(t) x(0)'\} a = CA^t P a = 0 \quad t = 0, 1, \dots$

$\Rightarrow Pa \in \bigcap_{t=0}^{\infty} \ker CA^t$ $\quad$ P nonsingular $\Rightarrow$ $X \cap (H^+)^\perp = 0 \Leftrightarrow \bigcap_{t=0}^{\infty} \ker CA^t = 0$

$\square$.

Thm 6.6.4: A stochastic realization is minimal (smallest dimension) iff its state space X is both observable and constructible

Let $\Lambda_t \triangleq E\{y(t) y(0)'\} = CA^{t-1}\bar{C}'$ for $t > 0$ $\Leftarrow$ Homework.

and
$$\Gamma = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \cdots \\ \Lambda_2 & \Lambda_3 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} C\bar{C}' & CA\bar{C}' & \cdots \\ CA\bar{C}' & CA^2\bar{C}' & \cdots \\ \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} \begin{bmatrix} \bar{C} \\ \bar{C}A' \\ \vdots \end{bmatrix}'$$

Rational spectral factorization

(C,A) is observable and $(\bar{C},A')$ is constructible iff $(A,C,\bar{C},\Lambda_0)$ in

$$\Phi_+(z) = C(zI-A)^{-1}\bar{C}' + \tfrac{1}{2}\Lambda_0$$

is a minimal (deterministic) realization of the rational function

$$\Phi_+(z) = \tfrac{1}{2}\Lambda_0 + \Lambda_1 z^{-1} + \Lambda_2 z^{-2} + \dots$$

Clearly $\Phi(z) = \Phi_+(z) + \Phi_+(z^{-1})' = \sum_{t=-\infty}^{\infty} \Lambda_t z^{-t}$

$$\left.\begin{array}{ll} A\text{ - stability matrix} \Rightarrow & \phi_+ \text{ has all poles in } |z|<1 \\ \text{Furthermore} & \Phi_+(e^{i\theta}) + \Phi_+(e^{-i\theta})' \geq 0 \quad \theta \in [-\pi,\pi]. \end{array}\right\} \begin{array}{l}\Phi_+ \text{ is} \\ \text{positive real.}\end{array}$$

Assume $W(z) = C(zI-A)^{-1}B + D$, then

$$W(z)W(z^{-1})' = [C(zI-A)^{-1}B+D][B'(z^{-1}I-A')^{-1}C'+D']$$

$$= C(zI-A)^{-1}BB'(z^{-1}I-A')^{-1}C' + C(zI-A)^{-1}BD' + DB'(z^{-1}I-A')^{-1}C' + DD'$$

Lyapunov eq. $\Rightarrow$ $BB' = P - APA'$

<u>Note:</u> $(zI-A)P(z^{-1}I-A') + (zI-A)PA' + AP(z^{-1}I-A') =$

$$= P + APA' \cancel{- zPA'} \cancel{- z^{-1}AP} + \cancel{zPA'} - APA' + \cancel{z^{-1}AP} - APA' = P - APA'$$

$\Rightarrow W(z)W(z^{-1})' = CPC' + CPA'(z^{-1}I-A')^{-1}C' + C(zI-A)^{-1}APC' + C(zI-A)^{-1}BD + DB'(z^{-1}I-A')^{-1}C' + DD'$

$\boxed{\text{Now use } \Lambda_0 = CPC' + DD' \text{ and } \bar{C} = CPA' + DB'}$

$$= \Lambda_0 + \bar{C}(z^{-1}I-A')^{-1}C' + C(zI-A)^{-1}\bar{C}' = \Phi_+(z) + \Phi_+(z^{-1})$$

$\therefore$ Φ_+ can be determined from W using the same (A,C) and then $\bar{C}, \Lambda_0$.

<u>Prop. 6.8.2</u>: Let W be an arbitrary rational analytic spectral factor of Φ. Then $\deg W \geq \deg \Phi_+$ where deg denotes McMillan degree.

<u>Proof.</u> (A,B,C,D) minimal realization of W $\Rightarrow$ $(A,C,\bar{C},\Lambda_0)$ realization of Φ_+ from above.

<u>Def 6.8.3</u>: The spectral factor W of Φ is <u>minimal</u> if $\deg W = \deg \Phi_+$.

<u>Ex</u>: W_- and W_+ are minimal.

The converse problem

Given $\Phi(z)$ s.t. $\Phi(e^{i\theta}) \geq 0 \ \forall \theta$, and Φ rational

- Determine (all) spectral factors $W(z)$.

First make the decomposition $\Phi(z) = \Phi_+(z) + \Phi_+(z^{-1})'$

By partial fraction decomposition so that Φ_+ has all poles in the open unit disc

and determine a minimal realization

$$\Phi_+(z) = C(zI-A)^{-1}\bar{C}' + \tfrac{1}{2}\Lambda_0.$$

$$\Rightarrow \Phi(z) = [C(zI-A)^{-1} \quad I] \begin{bmatrix} 0 & \bar{C}' \\ \bar{C} & \Lambda_0 \end{bmatrix} \begin{bmatrix} (z^{-1}I-A')^{-1}C' \\ I \end{bmatrix}$$

It is easy to check that

$$[C(zI-A)^{-1} \quad I] \begin{bmatrix} P-APA' & -APC' \\ -CPA' & -CPC' \end{bmatrix} \begin{bmatrix} (z^{-1}I-A')^{-1}C' \\ I \end{bmatrix} \equiv 0 \qquad (\forall z)$$

$$\Rightarrow \Phi(z) = [C(zI-A)^{-1} \quad I] \underbrace{\begin{bmatrix} P-APA' & \bar{C}'-APC' \\ \bar{C}-CPA' & \Lambda_0-CPC' \end{bmatrix}}_{=M(P)} \begin{bmatrix} (z^{-1}I-A')^{-1}C' \\ I \end{bmatrix} = 0.$$

If $M(P) \geq 0$, then it can be factored as $M(P) = \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}'$

$$\Rightarrow \Phi(z) = [C(zI-A)^{-1} \quad I] \begin{bmatrix} B \\ D \end{bmatrix} [B' \quad D'] \begin{bmatrix} (z^{-1}I-A')^{-1}C' \\ I \end{bmatrix} = W(z)W(z^{-1})'$$

where $W(z) = C(zI-A)^{-1}B + D$ is a minimal analytic spectral factor.

Let $\mathcal{P} \triangleq \{ P = P' \mid M(P) \geq 0 \}$.

Theorem 6.1: (The positive real lemma)

Let Φ_+ be a stable $m \times m$ transfer function with a minimal realization

$$\Phi_+(z) = C(zI-A)^{-1}\bar{C}' + \tfrac{1}{2}\Lambda_0$$

Let $M(P)$ be the linear map defined above.

Then the set $\mathcal{P}$ is nonempty iff Φ_+ is positive real

Finally, any $P \in \mathcal{P}$ is positive definite.

Proof: $\mathcal{P}$ nonempty $\Rightarrow \exists P: M(P) \geq 0 \quad \Rightarrow \phi_+(z) + \phi_+(z^{-1})' = W(z)W(z^{-1})' \geq 0 \quad |z|=1$.

Φ_+ positive real $\Rightarrow \exists$ spectral factor W_- corresponding to the forward predictor space.

$P_- = E\{x_-(0)x_-(0)'\}$ satisfies $M(P_-) \geq 0$

$\Rightarrow \mathcal{P}$ nonempty.

Theorem 6.8.4: There is a one-to-one correspondence between $P \in \mathcal{P}$ and minimal analytic spectral factors of Φ

$$P \in \mathcal{P} \quad \longleftrightarrow \quad W(z) = C(zI - A)^{-1}B + D \quad \text{where} \quad \begin{bmatrix} B \\ D \end{bmatrix}\begin{bmatrix} B \\ D \end{bmatrix}' = M(P)$$

is a unique (modulo orthogonal transformation) full-rank factor

Backward systems

Note: $M(P) = \begin{bmatrix} P & \bar{C}' \\ \bar{C} & \Lambda_0 \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} P [A' \; C'] = \text{Schur Compl}\left\{ \left[\begin{array}{cc|c} P & \bar{C}' & A \\ \bar{C} & \Lambda_0 & C \\ \hline A' & C' & P^{-1} \end{array}\right] \right\}$.

then $\text{Schur Compl}\left\{ \left[\begin{array}{c|cc} P & \bar{C}' & A \\ \hline \bar{C} & \Lambda_0 & C \\ A' & C' & P^{-1} \end{array}\right] \right\} = \begin{bmatrix} \Lambda_0 & C \\ C' & P^{-1} \end{bmatrix} - \begin{bmatrix} \bar{C} \\ A' \end{bmatrix} P^{-1} [\bar{C}' \; A]$

With $\bar{P} \triangleq P^{-1}$ let $\bar{M}(\bar{P}) = \begin{bmatrix} \bar{P} - A'\bar{P}A & C' - A'\bar{P}\bar{C}' \\ C - \bar{C}\bar{P}A & \Lambda_0 - \bar{C}\bar{P}\bar{C}' \end{bmatrix}$.

$A \to A'$ $\quad C \to \bar{C}$
$\bar{C} \to C$ $\quad P \to \bar{P}$

If $\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}' = \bar{M}(\bar{P})$ then $\bar{W}(z) = \bar{C}(z^{-1}I - A')^{-1}\bar{B} + \bar{D}$ is a coanalytic spectral factor

The Riccati inequality

By considering a Schur complement again we can formulate a Riccati inequality.

We will assume that $\Delta(P) = \Lambda_0 - CPC' > 0$ for all $P \in \mathcal{P}$.

Note that $\Delta(P) = DD'$ so $W(\infty) = D$ should be full rank for all spectral factors.

That is, there should be no zeros at infinity or zero for the spectral factors.

A process such that $\Delta(P) > 0 \;\; \forall P \in \mathcal{P}$ is called *regular*

Let $T \triangleq -(\bar{C}' - APC')\Delta(P)^{-1}$, then

$$\begin{bmatrix} I & T \\ 0 & I \end{bmatrix} M(P) \begin{bmatrix} I & 0 \\ T' & I \end{bmatrix} = \begin{bmatrix} -R(P) & 0 \\ 0 & \Delta(P) \end{bmatrix}$$

Riccati inequality

So $M(P) \geq 0$ iff $R(P) = APA' - P + (\bar{C}' - APC')\Delta(P)^{-1}(\bar{C}' - APC')' \leq 0$.

Moreover, $p = \operatorname{rank} M(P) = m + \operatorname{rank} R(P) \geq m$.

and if $R(P) = 0$, i.e. P solves the algebraic Riccati equation

then W_P is a square spectral factor $(m \times m)$

Let $\mathcal{P}_0 = \{P \in \mathcal{P}, R(P) = 0\} = \{P \mid P \text{ corresponds to a square spectral factor}\}$.

we show later $\quad = \{P \mid P \text{ corresponds to an internal state space}\}$