The forward and backward predictor space

A state space X s.t. $X \subset H$ is called internal.

Two examples are given in the following theorem

Theorem 6.7.1: Let y be p.n.d. m-dim (C,A) and $(\bar{C},A')$ observable; A stability matrix.

The predictor space $X_- \triangleq E^{H^-} H^+$ and

the backward predictor space $X_+ \triangleq E^{H^+} H^-$

are both state spaces of minimal stochastic realizations of y.

In fact, y has a stoch. realization $\Sigma_- \begin{cases} x_-(t+1) = A x_-(t) + B_- w_-(t) \\ y(t) = C x_-(t) + D_- w_-(t) \end{cases}$

With state space X_-, w_- is the forward innovation process of y, i.e. $H^-(w_-) = H^-$

Likewise, y has a backward stoch. real. $\Sigma_+ \begin{cases} \bar{x}_+(t-1) = A' \bar{x}_+(t) + \bar{B}_+ \bar{w}_+(t) \\ y(t) = \bar{C} \bar{x}_+(t) + \bar{D}_+ \bar{w}_+(t) \end{cases}$

with state space X_+, $\bar{w}_+$ is the backward innovation process of y, i.e. $H^+(\bar{w}_+) = H^+$

D_- and $\bar{D}_+$ are $m \times p$-matrices of full column rank, where p is the rank of the process y.

In particular, if y is full rank they are square and nonsingular

Assume that y is full rank $\Rightarrow$ D_-^{-1} exists.

$$\Rightarrow x_-(t+1) = A x_-(t) + B_- D_-^{-1} \left(y(t) - C x_-(t) \right)$$

and this can be interpreted as a steady state Kalman filter.

which is also a recursive form of Wiener filtering.

Kalman filtering

We will show that the steady-state Kalman filter is a minimal realization corresponding to the predictor space X_-.

Let $H_{[\tau,t]}(y) = \text{span}\{\alpha' y(k) \mid \alpha \in \mathbb{R}^m,\ k = \tau, \tau+1, \dots, t\}$

and $\hat{x}_k(t) = E^{H_{[\tau,t-1]}(y)} x_k(t) \qquad k = 1,2,\dots n. \qquad \hat{x}(\tau) = 0$ initialization

Let $\tilde{y}(t) = y(t) - C\hat{x}(t)$ be the innovation process.

Then $\hat{x}(t+1) = E^{H_{[\tau,t]}(y)} x(t+1) = \left(E^{H_{[\tau,t-1]}(y)} + E^{[\tilde{y}(t)]}\right) x(t+1)$

$$= A\hat{x}(t) + E\{x(t+1)\tilde{y}(t)'\}\left(E\{\tilde{y}(t)\tilde{y}(t)'\}\right)^{-1}\tilde{y}(t)$$

$$= A\hat{x}(t) + K(t-\tau)[y(t) - C\hat{x}(t)]$$

Proposition 6.9.1: The gain function K is given by $K(t) = (\bar{C}' - A\Pi(t)C')(\Delta(\Pi(t)))^{-1}$ where $\Pi(t)$ is the solution to the matrix Riccati equation

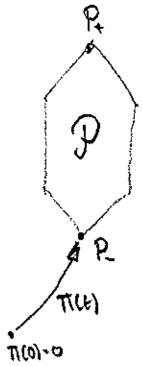
$$\Pi(t+1) = \Pi(t) + R(\Pi(t)) \quad , \ \Pi(0) = 0.$$

where $R(P) = APA' - P + (\bar{C}' - APC')\Delta(P)^{-1}(\bar{C}' - APC')'$, $\Delta(P) = \Lambda_0 - CPC'$ is assumed to be P.D.

Note: $K(t)$ depends only on $(A, C, \bar{C}, \Lambda_0)$, i.e. on Φ_+. Not on the particular spectral factor we started with.

Lemma 6.9.2: Let $\{\Pi(t)\}_{t\in\mathbb{Z}^+}$ be the solutions of the matrix Riccati equation

Then $P \geq \Pi(t+1) \geq \Pi(t)$ for all $P \in \mathcal{P}$ and $t = 0,1,2,\dots$



Theorem 6.9.3: $\Pi(t)$ tends monotonely to a limit $P_- \in \mathcal{P}_0$ as $t \to \infty$ which is the state covariance $P_- = E\{x_-(0)x_-(0)'\}$ of a stochastic realization whose state space is the predictor space X_-.

B_- and D_- are given by $\quad B_- = (\bar{C}' - AP_-C')\Delta(P_-)^{-1/2}$

$$D_- = \Delta(P_-)^{1/2}$$

Moreover $\quad \alpha' x_-(t) = E^{H_t^-}\alpha' x(t) \qquad \forall \alpha \in \mathbb{R}^n$

$$\begin{bmatrix} B_- \\ D_- \end{bmatrix}\begin{bmatrix} B_- \\ D_- \end{bmatrix}' = M(P_-)$$

and P_- is the minimum element of the family $\mathcal{P}$ in the sense that $P \geq P_- \quad \forall P \in \mathcal{P}$.

Dually $\bar{P} \geq \bar{P}_+ \quad \forall \bar{P} \in \bar{\mathcal{P}}$, where $\bar{P}_+$ is the limit of a backward Kalman filter cov. $\bar{\Pi}(t)$.

$\bar{P} = P^{-1} \quad \Rightarrow \quad P^{-1} \geq P_+^{-1} \quad \Rightarrow \quad P \leq P_+ \quad \forall P \in \mathcal{P}$

$\therefore \quad P_- \leq P \leq P_+ \quad \forall P \in \mathcal{P}$

$\mathcal{P}$ is convex and closed with a partial ordering

Chapter 7: The geometry of splitting subspaces.

Deterministic realization

Given a Hankel map $\mathcal{H}: \mathcal{U} \to \mathcal{Y}$ we want to find a realization

We want a state; so find a factorization
$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\mathcal{H}"] \arrow[dr, "\text{onto}"'] & & \mathcal{Y} \\ & X \arrow[ur, "1-1"'] & \end{tikzcd}$$

Solution: Let $\pi_H u = \{v \in \mathcal{U} \mid v \sim u\}$ be the canonical projection where $v \sim u$ if $\mathcal{H}v = \mathcal{H}u$, i.e. $v - u \in \ker \mathcal{H}$.

Let $X \triangleq \mathcal{U}/\ker\mathcal{H} = \{\pi_H u \mid u \in \mathcal{U}\}$ be the quotient space of all equivalence classes

Let φ be the operator $X \to \mathcal{Y}$ s.t. $\varphi(\pi_H u) = \mathcal{H}u$

$$\Rightarrow \begin{tikzcd} \mathcal{U} \arrow[rr, "\mathcal{H}"] \arrow[dr, "\pi_{\mathcal{H}}"'] & & \mathcal{Y} \\ & X \arrow[ur, "\varphi"'] & \end{tikzcd}$$ is a canonical factorization, i.e. $\pi_{\mathcal{H}}$ is onto, φ is 1-1.

Next we want to have an X that is easy to update with time.

Remember $\mathcal{Y} = \{y \mid y(t) = 0 \ \ t < 0\}$ which is invariant under the shift

$\sigma_t y(\tau) = y(t+\tau) \quad t \geq 0$, i.e. $\sigma_t \mathcal{Y} \subset \mathcal{Y}$.

We seek a restricted shift on X, i.e. an operator $\sigma_t(X): X \to X$ s.t the diagram
$$\begin{tikzcd} X \arrow[r, "\mathcal{O}"] \arrow[d, "\sigma_t(X)"'] & \mathcal{Y} \arrow[d, "\sigma_t"] \\ X \arrow[r, "\mathcal{O}"] & \mathcal{Y} \end{tikzcd}$$
commutes.

$\mathcal{O}$ - observability operator.

Let $\sigma_1(X) = \mathcal{O}^{\#} \sigma_1 \mathcal{O}(X)$. In Theorem 6.1.3 A is a matrix representation of this operator.

Furthermore $\sigma_t(X) = \sigma_1(X)^t$.

Now we want to do something similar for stochastic realization.

Splitting subspaces

Def. A subspace $X \subset \mathbb{H}$ is called a *splitting subspace* if $H^- \perp H^+ \mid X$

Remember $A \perp B \mid X$ if $\langle \alpha - E^X\alpha, \beta - E^X\beta \rangle = 0 \quad \forall \alpha \in A, \beta \in B$

Prop 2.4.2 $\Rightarrow$ alt. $E^A\beta = E^A E^X \beta \quad \forall \beta \in B$.

Here: $E^{H^+}\lambda = E^{H^+}E^X\lambda \quad \forall \lambda \in H^-$ i.e. a factorization of the Hankel operator $\mathcal{H} = E^{H^+}|_{H^-}$

$$\begin{tikzcd} H^- \arrow[rr, "\mathcal{H}"] \arrow[dr, "\mathcal{C}^*"'] & & H^+ \\ & X \arrow[ur, "\mathcal{O}"'] & \end{tikzcd}$$

where $\mathcal{O} \triangleq E^{H^+}|_X$ is the observability operator

$\mathcal{C}^* \triangleq E^X|_{H^-}$ is the adjoint of

$\mathcal{C} \triangleq E^{H^-}|_X$ the constructibility operator

$$\boxed{\mathcal{H} = \mathcal{O}\mathcal{C}^*}$$

Similarly

$$\begin{tikzcd} H^+ \arrow[rr, "\mathcal{H}^*"] \arrow[dr, "\mathcal{O}^*"'] & & H^- \\ & X \arrow[ur, "\mathcal{C}"'] & \end{tikzcd} \Rightarrow \boxed{\mathcal{H}^* = \mathcal{C}\mathcal{O}^*}$$

When are the factorizations canonical?

$\operatorname{Im} \mathcal{C}^* = E^X H^- \qquad \mathcal{C}^*: H^- \to X \quad \Rightarrow X = \ker \mathcal{C} \oplus \overline{\operatorname{Im}(\mathcal{C}^*)}$

$\overline{\operatorname{Im}\mathcal{C}^*} = X$ iff $\ker \mathcal{C} = 0$

on the other hand Lemma 2.2.6 $\Rightarrow A = E^A B \oplus (A \cap B^\perp)$

$$\Rightarrow X = \ker \mathcal{C} \oplus \underbrace{E^X H^-}_{\operatorname{Im}\mathcal{C}^*} = E^X H^- \oplus (X \cap (H^-)^\perp)$$

$\therefore \overline{\operatorname{Im} \mathcal{C}^*} = X$ iff $\ker \mathcal{C} = X \cap (H^-)^\perp = 0$

Similarly

$\overline{\operatorname{Im} \mathcal{O}^*} = X$ iff $\ker \mathcal{O} = X \cap (H^+)^\perp = 0$

Def 7.3.2: A splitting subspace X is *observable* if $X \cap (H^+)^\perp = 0$

constructible if $X \cap (H^-)^\perp = 0$.

Compare def 6.6.1

Thm 7.3.5: A splitting subspace is *minimal* iff it is both observable and constructible.

X is minimal if $X_1 \subset X$ is also a splitting subspace, then $X_1 = X$.