**Prop 7.3.1**: The predictor spaces $X_- \triangleq E^{H^-}H^+$ and $X_+ \triangleq E^{H^+}H^-$ are minimal splitting subspaces.

Infact, $X_-$ is the only minimal splitting subspace contained in $H^-$
and $X_+$ —————— " —————— $H^+$.

## Perpendicular Intersection

If $A \perp B \mid X$ and $A_1 \subset A$, $B_1 \subset B$ then $A_1 \perp B_1 \mid X$

But how much can we increase $A$ and $B$ and still satisfy the splitting property.

**Lemma 7.2.1**: Suppose $A \perp B \mid X$. Then
(i) $A \cap B \subset X$
(ii) $(A \vee X) \perp (B \vee X) \mid X$
(iii) $X = (A \vee X) \cap (B \vee X)$

So with $S \triangleq A \vee X$ and $\bar{S} \triangleq B \vee X$

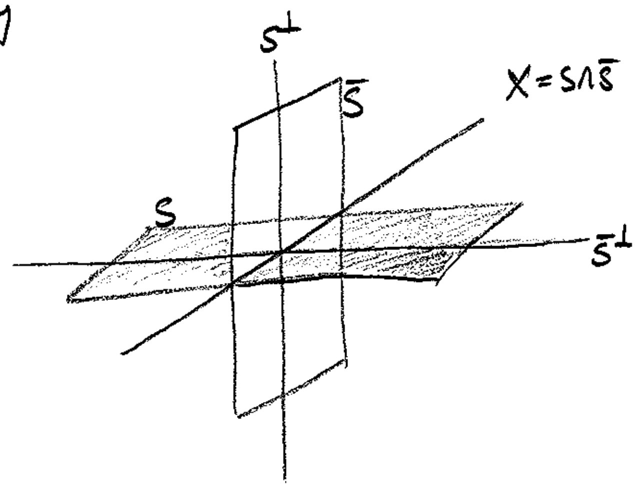
$A \perp B \mid X$ is equivalent to $S \perp \bar{S} \mid S \wedge \bar{S}$

**Def 7.2.3**: A pair $(S, \bar{S})$ of subspaces satisfying $S \perp \bar{S} \mid S \wedge \bar{S}$ are called <u>perpendicularly intersecting.</u>

**Prop 7.2.2**: The following are equivalent
(i) $S \perp \bar{S} \mid S \wedge \bar{S}$
(ii) $E^S \bar{S} = S \cap \bar{S}$
(iii) $E^{\bar{S}} S = S \cap \bar{S}$
(iv) $E^S \bar{S} = E^{\bar{S}} S$

**Theorem 7.2.4**: Let $S$ and $\bar{S}$ be subspaces such that $S \vee \bar{S} = \mathbb{H}$
Then the following conditions are equivalent.

(i) $S$ and $\bar{S}$ intersect perpendicularly
(ii) $\bar{S}^\perp \subset S$, or $S^\perp \subset \bar{S}$
(iii) $\mathbb{H} = \bar{S}^\perp \oplus (S \wedge \bar{S}) \oplus S^\perp$
(iv) $E^S$ and $E^{\bar{S}}$ commute



**Theorem 7.2.6:** Let $A$ and $B$ be subspaces s.t. $A \vee B = \mathbb{H}$ and $A \perp B \mid X$.

Let $A \subset S$ and $B \subset \bar{S}$.

Then $S \perp \bar{S} \mid X$ iff $S \subset A \vee X$ and $\bar{S} \subset B \vee X$.

If $S = A \vee X$ and $\bar{S} = B \vee X$, then $X = S \cap \bar{S}$

and $(S, \bar{S})$ intersects perpendicularly.

**Def:** A pair $(S, \bar{S})$ of perpendicularly intersecting subspaces such that $S \supset H^-$ and $\bar{S} \supset H^+$ is called a *scattering pair* of $X = S \cap \bar{S}$.

**Theorem 7.3.6:** A subspace $X \subset \mathbb{H}$ is a splitting subspace iff $X = S \cap \bar{S}$ for some scattering pair $(S, \bar{S})$.

Then $X = E^S \bar{S} = E^{\bar{S}} S$.

In particular $E^S \lambda = E^X \lambda \quad \forall \lambda \in \bar{S}$

and $E^{\bar{S}} \lambda = E^X \lambda \quad \forall \lambda \in S$.

A scattering pair is not unique for general $\mathbb{H}$.

If we take $\mathbb{H} = H = H^- \vee H^+$ then the scattering pair IS unique and $X \subset H$, i.e. $X$ is an *internal* splitting subspace.

**Prop. 7.3.7:** Suppose that $\mathbb{H} = H$. Then each splitting subspace $X$ has a unique scattering pair $(S, \bar{S})$, namely $S = H^- \vee X$ and $\bar{S} = H^+ \vee X$.

Proof: follows from Thm 7.2.6 and Thm 7.3.6.

## Markovian splitting subspaces

A subspace $X$ is a Markovian splitting subspace if $(H^- \vee X^-) \perp (H^+ \vee X^+) \mid X$

where $X^- \triangleq \overline{\text{span}}\{U^t X \mid t \le 0\}$ and $X^+ \triangleq \overline{\text{span}}\{U^t X \mid t \le 0\}$

Note: $X \subset X^-$, $X \subset X^+$

In particular: $H^- \perp H^+ \mid X$, i.e. $X$ is a splitting subspace.

$X^- \perp X^+ \mid X$, i.e. $X$ is Markovian.

Define the ambient space $\mathbb{H}_X \triangleq H \vee \overline{\text{span}}\{U^t X \mid t \in \mathbb{Z}\} = H^- \vee X^- \vee H^+ \vee X^+$.

Markovian representation $(\mathbb{H}_X, U, X)$.

smallest subspace of $\mathbb{H}$ which contains $H$ and $X$ and is invariant under $U, U^*$

**Thm 7.4.1** A splitting subspace $X$ is a Markovian splitting subspace iff it has a scattering pair $(S, \bar{S})$ such that $U^* S \subset S$, $U \bar{S} \subset \bar{S}$.

For each $X$ there is a unique such scattering pair contained in $\mathbb{H}_X$ and it is given by $S = H^- \vee X^-$, $\bar{S} = H^+ \vee X^+$

Moreover, $S \vee \bar{S} = \mathbb{H}_X$.

defines $X \sim (S, \bar{S})$ in $\mathbb{H}_X$

This is the smallest scattering pair representation of $X$.

$X = S \cap \bar{S} \subset S$ and $U^* X \subset U^* S \subset S \Rightarrow X^- \subset S$.

$H^- \subset S \Rightarrow H^- \vee X^- \subset S$ so the smallest $S$ satisfies with equality.

### Minimality

A Markovian splitting subspace is minimal if it contains no proper subspace which is also a Markovian splitting subspace.

Can minimality of $X$ be characterized in terms of $(S, \bar{S})$?

We know: $\mathbb{H}_X = S^\perp \oplus X \oplus \bar{S}^\perp$ (Thm 7.2.4. $\mathbb{H} = \bar{S}^\perp \oplus (S \cap \bar{S}) \oplus S^\perp$

$X$ small $\Rightarrow$ $S^\perp, \bar{S}^\perp$ large $\Rightarrow$ $S, \bar{S}$ small

**Lemma 7.4.2**: Let $X_1 \sim (S_1, \bar{S}_1)$ and $X_2 \sim (S_2, \bar{S}_2)$ be Markovian splitting subspaces.
Then $X_1 \subset X_2$ iff $S_1 \subset S_2$ and $\bar{S}_1 \subset \bar{S}_2$

How do we determine a minimal Markovian splitting subspace from a nonminimal one?

<u>Thm 7.4.3:</u> Let $X \sim (S, \bar{S})$ be a Markovian splitting subspace in $\mathbb{H}_X$ and set $\bar{S}_1 := H^+ \vee S^\perp \qquad S_1 = H^- \vee \bar{S}_1^\perp$ ($\perp$ in $\mathbb{H}_X$)

Then $X_1 \sim (S_1, \bar{S}_1)$ is a minimal Markovian splitting subspace such that $X_1 \subset X$.

By def. $S^\perp \subset \bar{S}$ and $H^+ \subset \bar{S} \Rightarrow H^+ \vee S^\perp = \bar{S}_1 \subset \bar{S}$

(Thm 7.3.6)

<u>Ex:</u> Let $S = H^-$ and $\bar{S} = H \Rightarrow X = S \cap \bar{S} = H^-$ is a splitting subspace.

$X$ is Markovian since $U^* S \subset S$ and $U\bar{S} \subset \bar{S}$ (Thm 7.4.1).

$\mathbb{H}_X = H \vee \overline{\text{span}}\{U^t X \mid t \in \mathbb{Z}\} = H$ so $X$ is internal. ($H^- \subset H$).

Let $\bar{S}_1 = H^+ \vee S^\perp = H^+ \vee (H^-)^\perp$

$$S_1 = H^- \vee \bar{S}_1^\perp = H^- \vee (H^+ \vee (H^-)^\perp)^\perp = H^- \vee \underbrace{(H^- \cap (H^+)^\perp)}_{\triangleq N^-} = H^-$$

Everything in the past orthogonal to the future

i.e with $N^- := H^- \cap (H^+)^\perp$ we have $\bar{S}_1 = (N^-)^\perp$ and $S_1 = H^-$

$$\Rightarrow X_1 = S_1 \cap \bar{S} = E^{S_1} \bar{S}_1 = E^{H^-}(H^+ \vee (H^-)^\perp) = E^{H^-} H^+ = X_-$$

<u>Prop 7.4.5:</u> The predictor space $X_- := E^{H^-} H^+$ is a minimal Markovian splitting subspace, and $X_- \sim (H^-, (N^-)^\perp)$.

<u>Prop 7.4.6:</u> The backward predictor space $X_+ := E^{H^+} H^-$ is a minimal Markovian splitting subspace, and $X_+ \sim ((N^+)^\perp, H^+)$ where $N^+ \triangleq H^+ \cap (H^-)^\perp$

<u>Cor. 7.4.7:</u> Every Markovian splitting subspace contains a minimal Markovian splitting subspace.

<u>Cor 7.4.8:</u> A Markovian splitting subspace $X \sim (S, \bar{S})$ is a minimal Markovian splitting subspace iff. $\bar{S} = H^+ \vee S^\perp$ and $S = H^- \vee \bar{S}^\perp$

<u>Thm 7.4.9:</u> A Markovian splitting subspace $X \sim (S, \bar{S})$ is

observable iff $\bar{S} = H^+ \vee S^\perp$ and

constructible iff $S = H^- \vee \bar{S}^\perp$.