Corollary 7.4.10: The Markovian splitting subspace $X \sim (S, \bar{S})$ is minimal iff it is both observable and constructible.

Corollary 7.4.11: A minimal Markovian splitting subspace is a minimal splitting subspace.

Corollary 7.4.12: A subspace X is an observable Markovian splitting subspace iff there is a subspace $S \supset H^-$, satisfying $U^* S \subset S$ such that $X = E^S H^+$.

It is a constructible Markovian splitting subspace iff there is a subspace $\bar{S} \supset H^+$, satisfying $U \bar{S} \subset \bar{S}$ such that $X = E^{\bar{S}} H^-$.

The Frame space

We know that $X_+ \sim ((N^+)^\perp, H^+) \Rightarrow X_+ = (N^+)^\perp \cap H^+$

$$\Rightarrow H^+ = (N^+ \oplus (N^+)^\perp) \cap H^+ = \underbrace{(N^+ \cap H^+)}_{= N^+ \text{ since } N^+ \subset H^+} \oplus ((N^+)^\perp \cap H^+) = N^+ \oplus X_+$$

similarly $H^- = N^- \oplus X_-$

Then $H = H^- \vee H^+ = (N^- \oplus X_-) \vee (N^+ \oplus X_+) = N^- \oplus H^\square \oplus N^+$ since $N^- \perp N^+$ and $X_- \subset H^- \perp N^+$, $X_+ \subset H^+ \perp N^-$

where $H^\square = X_- \vee X_+$ is the frame space

Prop 7.4.13: Let X be a splitting subspace. (not necessarily Markovian)

Then
$$E^{H^-} X = X_- \quad \text{iff} \quad X \perp N^-$$
$$E^{H^+} X = X_+ \quad \text{iff} \quad X \perp N^+.$$

Proof: $E^{H^-} X = E^{X_- \oplus N^-} X = \underbrace{E^{X_-} X}_{\subset X_-} + \underbrace{E^{N^-} X}_{= 0 \text{ if } X \perp N^-} \subset X_-$ iff $X \perp N^-$

$$X_- = E^{H^-} H^+ \underset{H^+ \subset \bar{S}}{\subset} E^{H^-} \bar{S} \underset{H^- \subset S}{=} E^{H^-} \underbrace{E^S \bar{S}}_{= X} = E^{H^-} X \quad \square$$

Cor. 7.4.14: Let X be a splitting subspace.

Then
$X \perp N^-$ if X is observable
$X \perp N^+$ if X is constructible

If X is minimal, it is orthogonal to both N^- and N^+.

Prop 7.4.15: The Frame space $H^\square$ is a Markovian splitting subspace, and $H^\square \sim ((N^+)^\perp, (N^-)^\perp)$.

Moreover, $X \wedge H \subset H^\square$ for all minimal splitting subspaces X.

Remember $H = N^- \oplus H^\square \oplus N^+ \;=\; \bar{S}^\perp \oplus (S \cap \bar{S}) \oplus S^\perp$

Now, clearly N^- and N^+ play no role in minimal state space construction
$X \perp N^-$, $X \perp N^+$

Note: $E^H X = \underbrace{E^{N^-} X}_{=0} + E^{H^\square} X + \underbrace{E^{N^+} X}_{=0} = E^{H^\square} X \subset H^\square = X_- \vee X_+$

for any minimal X

i.e. the smoothing estimate is a linear combination of the forward and backward predictors.

Minimality and dimension

Theorem 7.6.1: All minimal (Markovian or non-Markovian) splitting subspaces have the same dimension

Minimality in terms of subspace inclusion and dimension are the same.

Proof idea:

$$\begin{tikzcd} H^- \arrow[rr, "\mathcal{H}"] \arrow[dr, "C^*"] \arrow[ddr, bend right, "\tilde{C}^*"'] & & H^+ \\ & X \arrow[ur, "\mathcal{O}"] & \\ & \tilde{X} \arrow[uur, bend right, "\tilde{\mathcal{O}}"'] & \end{tikzcd}$$

X minimal $\iff$ $\overline{\operatorname{Im} C^*} = X$, $\overline{\operatorname{Im} \mathcal{O}^*} = X$.

If $\operatorname{Im} C^* = X$, then $\forall \xi \in X$

define $T: X \to \tilde{X}$ by $T\xi = \tilde{\mathcal{O}}^* \mathcal{O} \xi$.

If the Hankel op. $\mathcal{H} = E^{H^+}|_{H^-}$ has closed range

then it can be shown that T is invertible. $\Rightarrow$ $X, \tilde{X}$ has same dimension

Partial ordering of minimal splitting subspaces

Def: 7.7.1 Given two minimal splitting subspaces X_1 and X_2

Then $X_1 < X_2$ if $\|E^{X_1}\lambda\| \leq \|E^{X_2}\lambda\| \quad \forall \lambda \in H^+$.

$X_1 < X_2$ means that X_1 contains "less information" about the future than X_2

symmetrically:

Lemma 7.7.2: $X_1 < X_2$ iff $\|E^{X_2}\lambda\| \leq \|E^{X_1}\lambda\| \quad \forall \lambda \in H^-$.

$X_1 < X_2$ means that X_1 contains "more information" about the past than X_2

Theorem 7.7.3: The family of minimal splitting subspaces has a unique minimal element X_- and a unique maximal element X_+, i.e.

$$X_- \prec X \prec X_+$$

for all minimal X, and these are precisely the predictor spaces

$$X_- \triangleq E^{H^-}|_{H^+} \qquad \text{and} \qquad X_+ \triangleq E^{H^+}|_{H^-}$$

Proof idea: Take $\lambda \in H^+ = X_+ \oplus N^+ \Rightarrow E^X \lambda = E^X \mu$ where $\mu = E^{X_+}\lambda \in X_+$

since $X \perp N^+$ by minimality

Then $\|E^X \mu\| \leq \|\mu\| = \|E^{X_+}\mu\| \quad \forall \mu \in X_+$

i.e. $X \prec X_+$. The other inequality follows by a symmetric argument $\square$.

How does this ordering relate to the ordering of $\mathcal{P}$?

Define $X_1 \sim X_2$ if $X_1 \prec X_2$ and $X_2 \prec X_1$

and $\mathcal{X} \triangleq \{$the family of all equivalence classes of minimal splitting subspaces$\}$.

$\mathcal{X}_0 \subset \mathcal{X}$ the subset of internal splitting subspaces $(X \subset H)$

(if $X_1 \sim X_2$ and X_1 is internal $\Rightarrow X_1 = X_2$).

Next, lets define a uniform choice of bases for all minimal splitting subspaces X.

Let $(\xi_1^+, \xi_2^+, \dots, \xi_n^+)$ be an arbitrary basis in X_+

Lemma 7.7.4: The random variables $(\xi_1, \xi_2, \dots, \xi_n)$, where $\xi_k = E^X \xi_k^+ \quad k = 1, 2, \dots, n$, form a basis in X. (minimal).

For each X define $x = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}$ and $P = E\{xx'\}$. $\quad \mathcal{P} = \{ P = E\{xx'\} \mid \begin{smallmatrix} x \text{ uniform basis} \\ \text{for } X \in \mathcal{X} \end{smallmatrix} \}$

Proposition 7.7.5: There is a one-one correspondence between $\mathcal{X}$ and $\mathcal{P}$ which is order-preserving in the sense that

$$P_1 \leq P_2 \quad \text{iff} \quad X_1 \prec X_2.$$

Proof idea: $\lambda \in X_+ \Rightarrow \lambda = a' x_+ \Rightarrow E^X \lambda = E^X a' x_+ \overset{\xi_k = E^X \xi_k^+}{=} a' x$

$$\|E^{X_2}\lambda\|^2 - \|E^{X_1}\lambda\|^2 = a' E x_2 x_2' a - a' E x_1 x_1' a = a'(P_2 - P_1) a \geq 0.$$

Ordering and scattering pairs

Lemma 7.7.9: Let X_1 and X_2 be two minimal splitting subspaces, and suppose $(S_1, \bar{S}_1)$ is a scattering pair of X_1 and $(S_2, \bar{S}_2)$ is a scattering pair of X_2.

Then, $X_1 \prec X_2$ iff $\|E^{S_1}\lambda\| \le \|E^{S_2}\lambda\| \quad \forall \lambda \in H$

or equivalently $\|E^{\bar{S}_2}\lambda\| \le \|E^{\bar{S}_1}\lambda\| \quad \forall \lambda \in H.$

Cor. 7.7.10: Let X_1 and X_2 be equivalent minimal splitting subspaces. Then, if X_1 or X_2 is internal, $X_1 = X_2$.

Thm 7.7.11: Let $X_1 \sim (S_1, \bar{S}_1)$ and $X_2 \sim (S_2, \bar{S}_2)$ be minimal Markovian splitting subspaces. Then

(i) If $X_1, X_2 \in \mathcal{X}_0$, then $X_1 \prec X_2 \Leftrightarrow S_1 \subset S_2 \Leftrightarrow \bar{S}_2 \subset \bar{S}_1$

(ii) If $X_1 \in \mathcal{X}_0$, then $X_1 \prec X_2 \Leftrightarrow S_1 \subset S_2 \Leftrightarrow E^H \bar{S}_2 \subset \bar{S}_1$

(iii) If $X_2 \in \mathcal{X}_0$, then $X_1 \prec X_2 \Leftrightarrow E^H S_1 \subset S_2 \Leftrightarrow \bar{S}_2 \subset \bar{S}_1$