Chapter 8: Markovian representations.

A summary of the most important properties derived in the last chapter is given by

Thm 8.1.1: Given an m-dim. stationary process $\{y(t)\}_{t\in\mathbb{Z}}$, let $\mathbb{H} \supset H \triangleq H(y)$ be a Hilbert space of random variables with a shift U satisfying $Uy_k(t) = y_k(t+1)$, and let X be a subspace of $\mathbb{H}$ such that

$$\mathbb{H} = H \vee \overline{\operatorname{span}}\{U^t X \mid t\in\mathbb{Z}\}.$$

Then $(\mathbb{H}, U, X)$ is a Markovian representation of y iff $X = S\cap\bar{S}$ for some pair $(S,\bar{S})$ of subspaces of $\mathbb{H}$ such that

(i) $H^- \subset S$ and $H^+ \subset \bar{S}$

(ii) $U^*S \subset S$ and $U\bar{S} \subset \bar{S}$

(iii) $\mathbb{H} = \bar{S}^\perp \oplus (S\cap\bar{S}) \oplus S^\perp$ ($\perp$ denotes orthogonal complement in $\mathbb{H}$)

Moreover, the correspondence $X \leftrightarrow (S,\bar{S})$ is one-one.
In fact $S = H^- \vee X^-$ and $\bar{S} = H^+ \vee X^+$.

Finally, $(\mathbb{H}, U, X)$ is observable iff $\bar{S} = H^- \vee S^\perp$,
constructible iff $S = H^+ \vee \bar{S}^\perp$,

and minimal iff both conditions holds.

Forward realizations | **Backward realizations**

$$S \perp \bar{S} \mid X$$

$S \perp H^+ \mid X$ $\Leftarrow$ $S \perp \bar{S} \mid X$ $\Rightarrow$ $\bar{S} \perp H^- \mid X$

$$\begin{tikzcd} S \arrow[rr, "E^{H^+}|_S"] \arrow[dr, "E^X|_S"'] & & H^+ \\ & X \arrow[ur, "\mathcal{O} = E^{H^+}|_X"'] & \end{tikzcd}$$

$$E^A\beta = E^A E^X \beta \qquad \beta\in B$$

$$\begin{tikzcd} \bar{S} \arrow[rr, "E^{H^-}|_X"] \arrow[dr, "E^X|_{\bar{S}}"'] & & H^- \\ & X \arrow[ur, "C = E^{H^-}|_X"'] & \end{tikzcd}$$

$\mathcal{O}$ - observability operator

S - generalized past

$U^* S \subset S$

new info: $W = US \ominus S$

$X = S\cap\bar{S}$

$\Downarrow$ $\qquad$ $\Downarrow$

$X\subset S$ $\qquad$ $X \subset \bar{S}$

$\Downarrow$ $\qquad$ $\Downarrow$

$E^X|_S$ $\qquad$ $E^X|_{\bar{S}}$

are onto.

C - constructibility operator.

$\bar{S}$ - generalized future

$U\bar{S} \subset \bar{S}$

new info: $\bar{W} = \bar{S} \ominus U\bar{S}$

Let $Y \triangleq \{b'y(0) \mid b\in\mathbb{R}^m\}$.

Thm 8.1.2: Let $X \sim (S, \bar{S})$ be a Markovian splitting subspace, and let $W, \bar{W}$ and Y be as defined above. Then

$$\begin{cases} UX \subset X \oplus W \\ Y \subset X \oplus W \end{cases} \quad \text{and} \quad \begin{cases} U^* X \subset X \oplus (U^* \bar{W}) \\ U^* Y \subset X \oplus (U^* \bar{W}). \end{cases}$$

Proof: (left subspace inclusions)

$$UX = U(S \cap \bar{S}) = (US) \cap (U\bar{S}) = \{U\bar{S} \subset \bar{S}\} \subset (US) \cap \bar{S}$$

$$Y \subset (UH^-) \cap H^+ \subset (US) \cap \bar{S}$$

It remains to show that $(US) \cap \bar{S} = X \oplus W$.

$$W = US \ominus S \Rightarrow US = S \oplus W$$

$$(US) \cap \bar{S} = (S \oplus W) \cap \bar{S} = \left\{ \begin{matrix} S \perp W \\ \text{Prop. A.2.1} \end{matrix} \right\} = (S \cap \bar{S}) \oplus (W \cap \bar{S})$$

$$S \cap \bar{S} = X$$

$$W = US \ominus S \subset S^\perp \overset{\text{splitting property.}}{\subset} \bar{S} \Rightarrow W \cap \bar{S} = W$$

□.

$(W, \bar{W})$ satisfies the orthogonality relations $(U^j W) \perp (U^k W)$ for $j \neq k$.
$(U^j \bar{W}) \perp (U^k \bar{W})$

They are so called wandering subspaces.

Wold decomposition:

$$S = (U^{-1}W) \oplus (U^{-2}W) \oplus \dots \oplus (U^{-N}W) \oplus (U^{-N}S)$$

$$\bar{S} = \bar{W} \oplus (U\bar{W}) \oplus \dots + (U^{N-1}\bar{W}) \oplus (U^N \bar{S})$$

$$U^{-N}S \to S_{-\infty} \quad \text{and} \quad U^N \bar{S} \to \bar{S}_\infty \quad \text{as } N \to \infty.$$

We know that W and $\bar{W}$ are finite-dimensional (bounded by the multiplicity of $(\mathbb{H}, U, X)$) Thm 4.5.4 (Wold)

Let $\{\eta_1, \dots \eta_p\}$ be an orthonormal basis for W
$\{\bar{\eta}_1, \dots \bar{\eta}_p\}$ —— " —— $\bar{W}$.

Then $w(t) = \begin{bmatrix} U^t \eta_1 \\ \vdots \\ U^t \eta_p \end{bmatrix}$ and $\bar{w}(t) = \begin{bmatrix} U^t \bar{\eta}_1 \\ \vdots \\ U^t \bar{\eta}_p \end{bmatrix}$ are normalized white noise processes,

called the forward and backward generating processes.

Thm 8.1.3: Let $(\mathbb{H}, U, X)$ be a Markovian representation of multiplicity μ.

Then the wandering subspaces W and $\bar{W}$ have finite dimensions such that $p := \dim W \le \mu$ and $\bar{p} := \dim \bar{W} \le \mu$.

$\boxed{\text{Moreover, if } X \sim (S, \bar{S})}$ then $S = H^-(w) \oplus S_{-\infty}$

Forward

where $\{w(t)\}_{t\in\mathbb{Z}}$ is a p-dim. normalized white noise process ($E w(t) = 0$, $E w(t) w(s)' = I \delta_{ts}$)

such that $W := \{a' w(0) \mid a \in \mathbb{R}^p\}$

and $S_{-\infty}$ is a doubly invariant subspace. ($U^t S_{-\infty} \subset S_{-\infty}$ $t \in \mathbb{Z}$).

Backward

Similarly $\bar{S} = H^+(\bar{w}) \oplus \bar{S}_\infty$,

where $\{\bar{w}(t)\}_{t\in\mathbb{Z}}$ is a $\bar{p}$-dim normalized white noise process

such that $\bar{W} := \{a' \bar{w}(0) \mid a \in \mathbb{R}^{\bar{p}}\}$

and $\bar{S}_\infty$ is a doubly invariant subspace.

Finally $\mathbb{H} = H(w) \oplus S_{-\infty} = H(\bar{w}) \oplus \bar{S}_\infty$

and in particular $S^\perp = H^+(w)$ and $\bar{S}^\perp = H^-(\bar{w})$

Def 8.1.4: The Markovian representation $(\mathbb{H}, U, X)$ is <u>normal</u> if $S_{-\infty} = \bar{S}_\infty$ and <u>proper</u> if $S_{-\infty} = \bar{S}_\infty = 0$.

Markov semi-groups

$\Downarrow$

Cor 8.2.6: A finite-dimensional Markovian representation $(\mathbb{H}, U, X)$ is normal.

The forward and backward systems

Assume $(\mathbb{H}, U, X)$ is a finite-dimensional Markovian representation $(\dim X = n < \infty)$

Aim: Construct a forward (backward) stochastic system such that $X = \{a'x(0) \mid a \in \mathbb{R}^n\}$

Forward: Let $\{\xi_1, \xi_2, \dots, \xi_n\}$ be a basis for X.

Define
$$x(0) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \quad \text{and} \quad x(t) = U^t x(0) = \begin{bmatrix} U^t\xi_1 \\ U^t\xi_2 \\ \vdots \\ U^t\xi_n \end{bmatrix}$$

$$\Rightarrow \quad E\{x(t)x(t)'\} \overset{U \text{ unitary}}{=} E\{x(0)x(0)'\} = P \quad > 0 \quad \text{since } \{\xi_k\}_{k=1}^n \text{ is a basis.}$$

Thm 8.1.2

$$\hookrightarrow \begin{cases} UX \subset X \oplus W \\ Y \subset X \oplus W \end{cases} \Rightarrow \begin{cases} U\xi_i = \sum_{j=1}^n a_{ij}\xi_j + \sum_{j=1}^p b_{ij} w_j(0) & i = 1, 2, \dots, n. \\ y_i(0) = \sum_{j=1}^n c_{ij}\xi_j + \sum_{j=1}^p d_{ij} w_j(0) & i = 1, 2, \dots, m. \end{cases}$$

i.e.
$$\begin{cases} x(1) = Ax(0) + Bw(0) \\ y(0) = Cx(0) + Dw(0) \end{cases} \quad \text{Applying the shift } U^t \Rightarrow \quad \boxed{\Sigma \begin{cases} x(t+1) = Ax(t) + Bw(t) \\ y(t) = Cx(t) + Dw(t). \end{cases}}$$

Backward: Depending on the choice of basis for X, the $\bar{A}$ matrix of the backward system looks different. (State-space isomorphism $(T^{-1}AT, T^{-1}B, CT, D)$.).

Choose the "dual basis": Let $\{\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n\}$ be a basis for X s.t. $\langle \bar{\xi}_i, \xi_j \rangle = \delta_{ij}$.

So for
$$\bar{x}(-1) = \begin{bmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \\ \vdots \\ \bar{\xi}_n \end{bmatrix} \quad \text{and} \quad \bar{x}(t) = U^{t+1}\bar{x}(-1) \quad \text{we have} \quad E\{\bar{x}(t-1)x(t)'\} = I.$$

Since $\bar{x}(-1) = Tx(0)$ for some T. $\quad E\{\bar{x}(-1)x(0)'\} = T \cdot E\{x(0)x(0)'\} = I \quad \Rightarrow T = P^{-1}$

$$\Rightarrow \bar{P} = E\{\bar{x}(-1)\bar{x}(-1)'\} = P^{-1}E\{x(0)x(0)'\}P^{-1} = P^{-1}.$$

Thm 8.1.2

$$\hookrightarrow \begin{cases} U^*X \subset X \oplus (U^*\bar{W}) \\ U^*Y \subset X \oplus (U^*\bar{W}) \end{cases} \Rightarrow \begin{cases} U^*\bar{\xi}_i = \sum_{j=1}^n \bar{a}_{ij}\bar{\xi}_j + \sum_{j=1}^{\bar{p}} \bar{b}_{ij}\bar{w}_j(-1) & i = 1, 2, \dots, n. \\ y_i(-1) = \sum_{j=1}^n \bar{c}_{ij}\bar{\xi}_j + \sum_{j=1}^{\bar{p}} \bar{d}_{ij}\bar{w}_j(-1) & i = 1, 2, \dots, m. \end{cases}$$

i.e.
$$\begin{cases} \bar{x}(-2) = \bar{A}\bar{x}(-1) + \bar{B}\bar{w}(-1) \\ y(-1) = \bar{C}\bar{x}(-1) + \bar{D}\bar{w}(-1) \end{cases} \quad \text{Applying the shift } U^{t+1} \Rightarrow \quad \boxed{\bar{\Sigma} \begin{cases} \bar{x}(t-1) = \bar{A}\bar{x}(t) + \bar{B}\bar{w}(t) \\ y(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{w}(t). \end{cases}}$$

Theorem 8.3.1: Let $(\mathbb{H}, U, X)$ be a finite-dimensional Markovian representation and $n := \dim(X)$.
Then, to each choice of dual bases in X there is a pair of dual stochastic realizations, consisting of a forward system Σ and a backward system $\bar{\Sigma}$. They are unique modulo the choice of bases in the wandering subspaces W and $\bar{W}$, i.e. modulo multiplications of $\begin{bmatrix} B \\ D \end{bmatrix}$ and $\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}$ from the right by orthogonal transformations and which has the property

$$\{a' x(0) \mid a \in \mathbb{R}^n\} = X = \{a' \bar{x}(-1) \mid a \in \mathbb{R}^n\}$$

The forward and backward systems are connected via the relations

$$\bar{A} = A', \qquad \bar{C} = CPA' + DB'$$

and

$$\bar{x}(t-1) = \bar{P}^{-1} x(t), \qquad \bar{P} = P^{-1}$$

where

$$P = E\{x(t) x(t)'\} \quad \text{and} \quad \bar{P} = E\{\bar{x}(t) \bar{x}(t)'\} \quad \forall t \in \mathbb{Z}$$

Moreover, the splitting subspace X is observable

iff $\bigcap_{t=0}^{\infty} \ker CA^t = 0$ $\quad$ (C, A) completely observable

and constructible

iff $\bigcap_{t=0}^{\infty} \ker \bar{C}(A')^t = 0$ $\quad$ $(\bar{C}, A')$ completely observable

Finally, the Markovian representation is minimal iff both (C, A) and $(\bar{C}, A')$ are observable.

Prop. 8.3.2: A purely nondeterministic, stationary, vector process $\{y(t)\}_{t \in \mathbb{Z}}$ has a rational spectral density iff it has a finite-dimensional Markovian representation $(\mathbb{H}, U, X)$.

The Markov semigroup

(7.5) **Theorem 7.5.1:** Let $X \sim (S, \bar{S})$ be a Markovian splitting subspace.

Then, for $t = 0, 1, 2,$ the diagrams

$$\begin{tikzcd} H^+ \arrow[r, "\mathcal{O}^*"] \arrow[d, "U^t"'] & X \arrow[d, "U_t(X)"] \\ H^+ \arrow[r, "\mathcal{O}^*"] & X \end{tikzcd} \qquad \begin{tikzcd} H^- \arrow[r, "\mathcal{C}^*"] \arrow[d, "(U^*)^t"'] & X \arrow[d, "U_t(X)^*"] \\ H^- \arrow[r, "\mathcal{C}^*"] & X \end{tikzcd}$$

commute.

Here: $\mathcal{O} = E^{H^+}|_X$ is the observability operator

$\mathcal{C} = E^{H^-}|_X$ is the constructibility operator

$U_t(X) = E^X U^t|_X$ is the restricted shift on X.

Moreover, the restricted shift satisfies the semigroup property

$$U_s(X) U_t(X) = U_{s+t}(X)$$

For each $\xi \in X$ and $t = 0, 1, 2, \dots$

$$E^S U^t \xi = U_t(X)\xi$$
$$E^{\bar{S}} U^{-t} \xi = U_t(X)^* \xi$$

Theorem 8.2.1: The semigroup $U_t(X)$ tends strongly to zero as $t \to \infty$ iff

$$S_{-\infty} \triangleq \bigcap_{t=-\infty}^{0} U^t S = 0.$$

$U_t(X)^* \to 0$ as $t \to \infty$ iff $\bar{S}_\infty = 0$

$(\mathbb{H}, U, X)$ is proper $\Leftrightarrow$ both $U_t(X)$ and $U_t(X)^*$ tends strongly to zero as $t \to \infty$.

Thm 8.2.3: The Markovian splitting subspace X admits a unique decomposition

$X = X_0 \oplus X_\infty$ such that $U(X)|_{X_\infty}$ is unitary

$U(X)|_{X_0}$ is completely nonunitary.

Moreover, $X_\infty = S_{-\infty} \cap \bar{S}_\infty$.

Thm 8.2.7: Let $X = X_0 \oplus X_\infty$, then each of the following is sufficient for $(\mathbb{H}, U, X)$ to be normal.

1. The intersection of the spectrum of $U(X)|_{X_0}$ with the unit circle has Lebesgue measure zero.

2. There is a nontrivial $\varphi \in H^\infty$ s.t. $\varphi(U(X)|_{X_0}) = 0$.

jmfr $\chi_A(A) = 0$