Homework 1

Please hand in the homework before, or on, the 20 of March. If you get stuck on something do not hesitate to ask me.

1 The definition of a (Hilbert-)adjoint operator is as follows.
If $T : H_1 \to H_2$ is a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the Hilbert-adjoint operator T^* *of T is the operator $T^* : H_2 \to H_1$ such that for all $x \in H_1$ and $y \in H_2$,*

$$\langle Tx, y\rangle = \langle x, T^* y\rangle.$$

We have claimed that the adjoint of the shift operator $U_t : \mathbf{H} \to \mathbf{H}$ is $U_t^* = U_t^{-1}$, and this is easy to verify from the definition. The adjoint of the adjoint U_t^* is then U_t.

If we instead consider this operator U_t^* on the invariant subspace $\mathbf{X}^-$, i.e. $U_t^* : \mathbf{X}^- \to \mathbf{X}^-$, it's adjoint is no longer U_t, why ?

On page 21 of the book, the compressed right shift

$$\begin{array}{rcl} T_t : \mathbf{X}^- & \to & \mathbf{X}^- \\ \xi & \mapsto & \mathrm{E}^{\mathbf{X}^-} U_t \xi \end{array}, \quad t \geq 0,$$

is introduced and it is claimed that this is the adjoint of U_t^* on the invariant subspace $\mathbf{X}^-$. Verify this from the definition.

The purpose of considering the compressed right shift is to characterize the Markov property in terms of invariance for the operator T as in Proposition 2.6.1.

2 We will consider a discrete time version of Example 3.6.2. So let x, y be a joint stationary process generated by the linear stochastic system

$$\begin{cases} x(t+1) & = & Ax(t) + Bw(t), \\ y(t) & = & Cx(t) + Dw(t), \end{cases}$$

where w is a normalized white noise process, i.e. it's spectral distribution function F_w satisfies $dF_w(\theta) = \frac{1}{2\pi} d\theta$. Assume that A is a stability matrix so the sum

$$x(t) = \sum_{k=0}^{\infty} A^k B w(t-k-1)$$

converges. (and it will not matter how x was initiated)

Corresponding to the white noise w, there is a spectral measure $d\hat{w}$ such that

$$w(t) = \int_{-\pi}^{\pi} e^{i\theta t} d\hat{w}(\theta).$$

According to Theorem 3.3.1 there is an $\hat{f}$ such that $x(0)$ has a spectral representation

$$x(0) = \int_{-\pi}^{\pi} \hat{f}(e^{i\theta}) d\hat{w}(\theta) = \mathcal{I}_{\hat{w}}(\hat{f}),$$

and then

$$x(t) = \int_{-\pi}^{\pi} e^{it\theta} \hat{f}(e^{i\theta}) d\hat{w}(\theta) = \mathcal{I}_{\hat{w}}(e^{it\theta} \hat{f}),$$

Determine $\hat{f}$.

Since $x(0)$ is spanned by the white noise, it follows from Theorem 3.5.1 that

$$x(0) = \sum_{s=-\infty}^{\infty} f(-s) w(s) = \mathcal{I}_w(f),$$

where $f \in \ell_m^2$, and

$$x(t) = \sum_{s=-\infty}^{\infty} f(t-s) w(s) = \mathcal{I}_w(T^t f).$$

Determine f, and show that $\hat{f}$ is the Fourier transform of f.

Now we will consider the process y. First note that y has a spectral representation in terms of an orthogonal increment process $\hat{y}$ such that $d\hat{y}(\theta) = W(e^{i\theta}) d\hat{w}(\theta)$, i.e.

$$y(t) = \int_{-\pi}^{\pi} e^{it\theta} d\hat{y}(\theta) = \mathcal{I}_{\hat{y}}(e^{it\theta}) = \int_{-\pi}^{\pi} e^{it\theta} W(e^{i\theta}) d\hat{w}(\theta) = \mathcal{I}_{\hat{w}}(e^{it\theta} W),$$

for some function W. Determine W.

Finally, use Theorem 3.1.3 and the spectral representation of y to derive an expression for

$$\Lambda(\tau) = \mathrm{E}\{y(t)\overline{y(t+\tau)}\},$$

and determine the spectral distribution function F_y of y. (we can assume that y is scalar)

3 Assume that x and y are two jointly stationary second order processes, with spectral density given by

$$\Phi(z) = \begin{bmatrix} \Phi_x & \Phi_{xy} \\ \Phi_{yx} & \Phi_y \end{bmatrix} = \begin{bmatrix} \frac{200+40(z+z^{-1})}{3-(z+z^{-1})} & \frac{1}{z+1/2} \\ \frac{1}{z^{-1}+1/2} & e^{z+z^{-1}} \end{bmatrix}.$$

Determine a cascade implementation of the causal Wiener filter for estimating x based on the past values of y.

Answer by giving W and $\hat{F}$

Hint: If w is a process determined by filtering y with W^{-1}, then $\Phi_{xw} = \Phi_{xy}W^{-1}$.

Would the result be different if we exchange the expressions for Φ_{xy} and Φ_{yx} ?

4 Consider the forward stochastic system

$$\Sigma \quad \begin{cases} x(t+1) &= Ax(t) + Bw(t) \\ y(t) &= Cx(t) + Dw(t) \end{cases},$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 2/3 \end{bmatrix}, D = 0;$$

and $w(t)$ is a normalized white noise.

(a) Determine the transfer function $W(z)$ for Σ.
Is Σ a minimal (deterministic) realization for W ?

(b) Determine the backward system $\bar{\Sigma}$ corresponding to the same state space $\mathbf{X}_t$ as for Σ.

(c) Determine the transfer function $\bar{W}(z)$ for $\bar{\Sigma}$.
Is $\bar{\Sigma}$ a minimal (deterministic) realization for $\bar{W}$?

(d) Is Σ a minimal (stochastic) realization for Σ ?

(e) Determine the structural function K of $(\Sigma, \bar{\Sigma})$.
What are the poles and zeros of K ?