Lectures 11-12
Applications in Control and Communications

Mikael Johansson
S3/Automatic Control
mikaelj@s3.kth.se

Anders Forsgren
Math/OptSyst
andersf@kth.se

Disposition

- SDP basics
- Applications in control
- Applications in communications (next week)

Linear matrix inequalities

An inequality on the form

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0$$

where F_i are symmetric matrices, and x_1, ..., x_n are scalar unknowns

Note: F_i can be complex-valued Hermetian matrices

Some standard problems

SDPs (convex)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0 \end{array}$$

Generalized eigenvalue problems (quasi-convex)

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \lambda B(x) - A(x) > 0, \quad B(x) > 0, \quad C(x) > 0 \end{array}$$

Max-det problems (convex)

$$\begin{array}{ll} \text{minimize} & \log\det A(x)^{-1} \\ \text{subject to} & A(x) > 0 \qquad B(x) > 0 \end{array}$$

SDP dual

Recall that the dual to the SDP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0 \end{array}$$

is the SDP

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr}(F_0 Z) \\ \text{subject to} & \mathbf{Tr}(F_i Z) = c_i \\ & Z \geq 0 \end{array}$$

Key techniques

Some key tricks and techniques for LMIs in control
1. S-procedure
2. Schur complements
3. Congruence transforms
4. Finsler's lemma

We will cover 1-3 in some detail.

S-procedure

Does it hold that

$$f(x) \geq 0 \qquad \text{for all } x \,:\, g(x) \leq 0$$

Equivalently: do we have

$$\inf_{x \in C,\, g(x) \leq 0} f(x) \geq 0$$

By weak and strong duality:

- yes, if there exists $\lambda \geq 0$ with

$$f(x) + \lambda^T g(x) \geq 0 \qquad \text{for all } x \in C$$

- Suppose C, f, g, convex and $\exists x_0 \in C$ with $g(x_0)<0$, then above conditions are necessary and sufficient

S-procedure cont'd

Nontrivial result without convexity:

Let f,g $\mathbb{R}^n \mapsto \mathbb{R}$ be quadratic functions such that $\exists\, x_0$ with $g(x_0)<0$, then the answer is yes if and only if there exists $\lambda \geq 0$ such that

$$f(x) + \lambda g(x) \geq 0 \qquad \text{for all } x \in C$$

Let $f(x)=x^TFx$ and $g(x)=x^TGx$, then the above condition reads

$$F + \lambda G \geq 0$$

an LMI in the variable λ

Example: quadratic confinement

Let $\mathcal{Q}_i = \{ x \mid q_i(x) = x^TA_ix+2b_i^Tx+c_i \geq 0\}$. Does $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$?

Yes, if there is $\lambda \geq 0$ such that

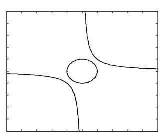
$$\begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} - \lambda \begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} \geq 0$$

Example: does: $\{x : -x_1^2-x_2^2+1 \geq 0\} \subseteq \{x : -2x_1x_2+2 \geq 0\}$

Yes, since

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \geq 0$$

for, e.g., $\lambda=1.5$



One (of many) extensions

We have

$$C + X^TB + B^TX + X^TAX \geq 0 \text{ for all } X \text{ with } I - X^TDX \geq 0$$

If and only if there exists $\lambda \geq 0$ such that

$$\begin{bmatrix} C & B^T \\ B & A \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \\ 0 & -D \end{bmatrix} \geq 0$$

Schur complement

The condition

$$M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$$

holds if and only if

$$Q > 0 \qquad R - S^TQ^{-1}S > 0$$

Schur complement proof

The condition

$$M > 0$$

is equivalent to

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}^T = x^TQx + 2x^TSy + y^TRy > 0 \qquad \text{for all } (x,y)$$

i.e.

$$\inf_x x^TQx + 2x^TSy + y^TRy > 0 \qquad \text{for all } y$$

The minimum is attained for $x=-Q^{-1}Sy$, so this is equivalent to

$$y^T[R - S^TQ^{-1}S]y > 0 \qquad \text{for all } y$$

Congruence transforms

The condition

$$M > 0$$

holds if and only if

$$T^T M T > 0$$

for some nonsingular matrix T.

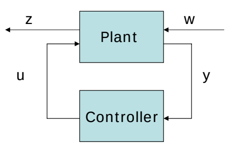
Example. Shur complement proved by congruence transform with

$$T = \begin{bmatrix} I & -Q^{-1}S \\ 0 & I \end{bmatrix}$$

Disposition

- SDP basics
- Applications in control

Control system setup



Design controller that maps measurements y into controls u

- Stability of closed-loop system
- Performance (e.g., surpression of disturbance w)
- Many different control structures; here full state feedback

Stability analysis via Lyapunov functions

The system $\dot{x} = Ax$ is exponentially stable iff there exists P with

$$P > 0 \qquad A^T P + PA < 0$$

In particular, there exists ε>0 such that all trajectories satsify

$$\|x(t)\| \leq \|x(0)\| \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\varepsilon t}$$

Note: multiple LMIs still LMI, Lyapunov inequalities equivalent to

$$\begin{bmatrix} P & 0 \\ 0 & -A^T P + PA \end{bmatrix} < 0$$

Lyapunov proof

The Lyapunov inequalities imply that there is ε>0 such that

$$A^T P + PA + \epsilon I \leq 0$$

Let $V(x)=x^T Px$. Then, for all $t \in R$,

$$\frac{d}{dt} V(x(t)) + \epsilon V(x(t)) = x^T(t)[A^T P + PA]x(t) + \epsilon x^T P x$$
$$= x^T[A^T P + PA + \epsilon P]x \leq 0$$

After integration, this yields, for all $t \geq t_0$

$$x^T(t)Px(t) \leq x^T(t_0)Px(t_0)e^{-\epsilon t}$$

The result now follows from $\lambda_{\min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{\max}(P)\|x\|^2$

Simultaneous Lyapunov function

Consider the piecewise linear system

$$\dot{x} = A_i x \qquad \text{for } x \in X_i$$

All trajectories of the system go to zero exponentially if there is P with

$$P > 0$$
$$A_i^T P + PA_i < 0, \qquad \forall i$$

Still LMI conditions.

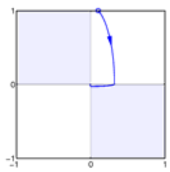
Challenge: derive less conservative conditions when

$$X_i = \left\{ x \mid x^T Q_i x \geq 0 \right\}$$

Example:

Consider the piecewise linear system

$$\dot{x} = \begin{cases} \begin{bmatrix} -0.1 & 1 \\ -10 & -0.1 \end{bmatrix} & \text{if } x_1 x_2 \geq 0 \\ \begin{bmatrix} -0.1 & 10 \\ -1 & -0.1 \end{bmatrix} & \text{if } x_1 x_2 \leq 0 \end{cases}$$



No common Lyapunov function unless partition information exploited!

Simultaneous stabilization

Given the system

$$\dot{x} = A_i x + B_i u \qquad \text{for } x \in X_i$$

Is there a state feedback u=Lx, which stabilizes the system?

$$\dot{x} = A_i x + B_i u \qquad \text{for } x \in X_i$$

Lyapunov conditions read

$$P > 0 \quad (A_i + B_i L)^T P + P(A_i + B_i L) < 0$$

Not LMIs! Considering congruence transform with $Q=P^{-1}$ gives

$$Q > 0 \quad QA^T + AQ + QL^T B_i^T + B_i L Q < 0$$

LMI conditions in (Q,Y) where $Y = LQ$

No common Lyapunov function

For given A_i, how would you prove that there is no solution P to

$$P > 0$$
$$A_i^T P + P A_i < 0, \qquad \forall i$$

Claim: if there exist $R_i > 0$ such that

$$\sum_i R_i A_i^T + A_i R_i > 0$$

then there is no P satisfying the Lyapunov inequalities above.

Proof:

Example: L_2 gain analysis

Closed loop linear system with disturbance w and output y

$$\dot{x} = Ax + Bw \qquad y = Cx \qquad x(0) = 0$$

has L_2 gain γ if, for all (square integrable) w and all $t \geq 0$,

$$\int_{\tau=0}^{t} y(\tau)^T y(\tau)\, d\tau \leq \gamma^2 \int_{\tau=0}^{t} w(\tau)^T w(\tau)\, d\tau$$

Alternative characterization: non-negative storage function $V(x) \geq 0$

$$\frac{\partial V}{\partial x}(Ax + Bw) \leq \gamma^2 w^T w - y^T y$$

How to find storage function? How to find the best one (smallest γ)?

L_2 gain analysis via convex optimization

Fact: for linear systems, sufficient to consider quadratic V(x)

$$V(x) = x^T P x$$

Gain conditions translate to *linear matrix inequalities* (in P, γ^2)

$$P > 0$$
$$\begin{bmatrix} A^T P + PA + C^T C & PB \\ BP & -\gamma^2 I \end{bmatrix} < 0$$

Convex conditions: can find optimal P and smallest γ efficiently!

State feedback design

Find state feedback law $u = Lx$ that minimizes L_2 induced gain $w \to y$

$$\dot{x} = Ax + B_u u + Bw$$
$$y = Cx$$

Claim: If there exists Q, γ and Y such that

$$Q > 0$$
$$\begin{bmatrix} AQ + QA^T + B_u Y + Y^T B_u^T + BB^T & QC^T \\ CQ & -\gamma^2 I \end{bmatrix} < 0$$

then, the state feedback $u=YQ^{-1}x$ makes the induced L_2-gain of the closed-loop system less than γ

Proof:

Much more

The framework has many extensions:

- H_2 and H_∞ designs
- Constraints on closed loop pole locations
- Dynamic output feedback
- Systems in discrete-time
- Uncertain systems
- Nonlinear and parameter-varying systems
- Hybrid systems

⋮