KTH Mathematics

2E5295/5B5749 Convex optimization with engineering applications

Lecture 2

Convexity

Global minimizer and local minimizer

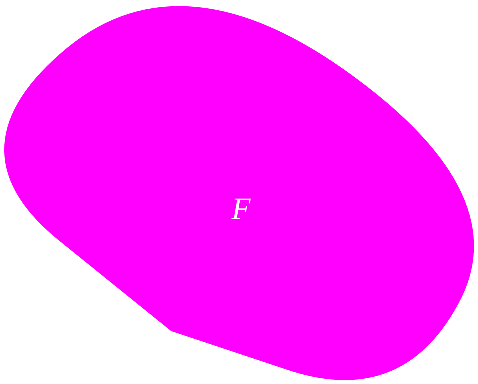
$$(P) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in S. \end{array}$$

Definition. *A point* $x^* \in S$ *is a* global minimizer *to* (P) *if* $f(x^*) \le f(x)$ *for all* $x \in S$.

Definition. *A point* $x^* \in S$ *is a* local minimizer *to* (P) *if there is an* $\epsilon > 0$ *such that* $f(x^*) \le f(x)$ *for all* $x \in S$ *that satisfy* $\|x - x^*\| < \epsilon$.

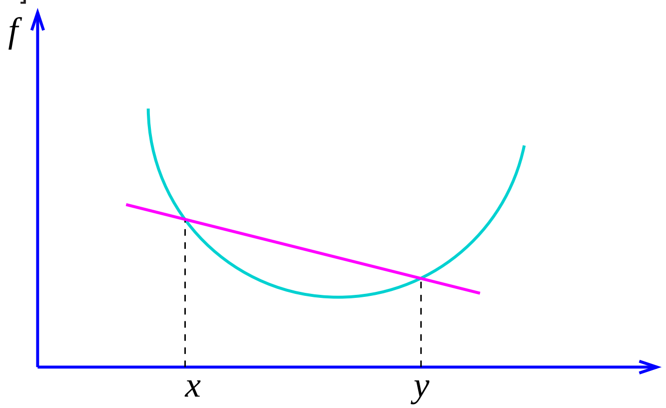
Convex set

Definition. *A set* S *is a* convex set *if* $(1-\alpha)x + \alpha y \in S$ *for all* $x \in S$, $y \in S$ *and* $\alpha \in [0,1]$.



Convex function

Definition. *If* S *is a convex set, then* $f : S \to \mathbb{R}$ *is a* convex function *on* S *if* $f((1-\alpha)x + \alpha y) \le (1-\alpha)f(x) + \alpha f(y)$ *for all* $x \in S$, $y \in S$ *and* $\alpha \in [0,1]$.



A function f is *concave* on a convex set S if $-f$ is convex on S.

Convexity gives global minimality

$$(P) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in S. \end{array}$$

Theorem. *Assume that S is a convex set and f is a convex function on S. If x^* is a local minimizer to (P), then it is also a global minimizer to (P).*

Proof. Assume that $x^* \in S$ is not a global minimizer. Then there is an $y \in S$ such that $f(y) < f(x^*)$. Convexity gives

$$(1-\alpha)x + \alpha y \in S \quad \text{and} \quad f((1-\alpha)x + \alpha y) < f(x)$$

for $\alpha \in (0,1)$. Hence, x^* is not a local minimizer. $\square$

Operations that preserve convexity

Proposition. *If S_1 and S_2 are convex sets, then $S_1 \cap S_2$ is convex.*

Proposition. *If f_1 and f_2 are convex functions on a convex set S, then $f_1 + f_2$ is convex on S.*

Proposition. *If f_1 and f_2 are convex functions on a convex set S, then $\max\{f_1, f_2\}$ is convex on S.*

Proposition. *If S is a convex set and $f(x) = Ax + b$, then $f(S) = \{f(x) : x \in S\}$, is a convex set.*

Proposition. *If S is a convex set and $f(x) = Ax + b$, then $f^{-1}(S) = \{x : f(x) \in S\}$, is a convex set.*

Proposition. *If $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is a convex set, then*

$$\{x_1 \in \mathbb{R}^m : (x_1, x_2) \in S \quad \text{for some } x_2 \in \mathbb{R}^n \quad \}$$

is a convex set.

Operations that preserve convexity, cont.

Proposition. *Let $S = \mathbb{R}^n \times \mathbb{R}_{++}$ and let $f : S \to \mathbb{R}$ be defined by $f(z,t) = \frac{1}{t} z$. Then $\{f(z,t) : (z,t) \in S\}$ is a convex set.*

Proposition. *For $g : \mathbb{R}^n \to \mathbb{R}^{m+1}$, defined by*

$$g(x) = \begin{pmatrix} A \\ c^T \end{pmatrix} x + \begin{pmatrix} b \\ d \end{pmatrix},$$

let $S = \{x \in \mathbb{R}^n : c^T x + d > 0\}$, and let

$$f(x) = \frac{1}{c^T x + d} \left(Ax + b \right).$$

Then, $\{f(x) : x \in S\}$ is a convex set.

Convexity and differentiability

Proposition. *Let $S = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i \in \mathcal{I},\ g_i(x) = 0,\ i \in \mathcal{E}\}$. Then S is a convex set if g_i, $i \in \mathcal{I}$, are concave functions on $\mathbb{R}^n$ and g_i, $i \in \mathcal{E}$, are affine functions on $\mathbb{R}^n$.*

Proposition. *Assume that S is a convex set and assume that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $f \in C^1$. Then f is a* convex function *on S if and only if $f(y) \geq f(x) + \nabla f(x)^T (y - x)$ for all $x \in S$, $y \in S$.*

Proposition. *Assume that S is a convex set and assume that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $f \in C^2$. Then f is a* convex function *on S if and only if $(y-x)^T \nabla^2 f(x)(y-x) \geq 0$ for all $x \in S$, $y \in S$.*

Corollary. *Assume that S is a convex set and assume that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $f \in C^2$. Then f is a* convex function *on S if $\nabla^2 f(x) \succeq 0$ for all $x \in S$.*

Jensen's inequality

Theorem. *If S is a convex set, then $f : S \to I\!R$ is a* convex function *on S if and only if*

$$f\left(\sum_{i=1}^{m} \alpha_i x_i\right) \le \sum_{i=1}^{m} \alpha_i f(x_i),$$

for all integers $m \ge 2$, for all $\alpha \in I\!R^m$ such that $\alpha_i \ge 0$, $i = 1, \dots, m$, $\sum_{i=1}^{m} \alpha_i = 1$, and for all $x_i \in S$, $i = 1, \dots, m$.

Generalization of the definition (for $m = 2$).

Other convexity concepts

Convex functions may be defined in terms of convex sets. The *epigraph* of a function $f : S \to I\!R$ is defined as

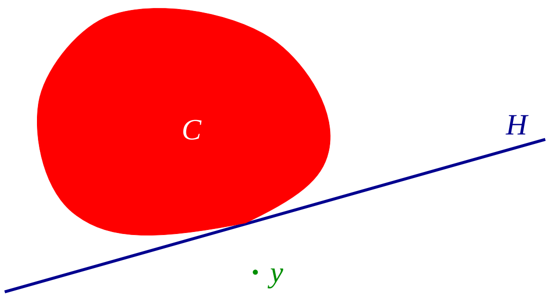
$$\text{epi}(f) = \{(x, t) : x \in S, f(x) \le t\}.$$

Proposition. *Let S be a convex set and let $f : S \to I\!R$. Then, f is a convex function on S if and only if* epi(f) *is a convex set.*

Convexity is a wide area. There are other concepts such as *quasi-convexity* and *pseudo-convexity*. We stick to convexity in this course.

The separating hyperplane theorem

Theorem. *Let S be a closed convex set in $I\!R^n$ and let $y \in I\!R^n$. If $y \notin S$, then there is a hyperplane H, with $H = \{x \in I\!R^n : a^T x = b\}$, such that $a^T x \ge b$ for all $x \in S$ and $a^T y < b$.*



Proof of the separating hyperplane theorem

Proof. Consider the problem $\underset{x \in S}{\text{minimize}} \|x - y\|_2^2$.

Since S is closed and $y \notin S$, this problem has an optimal solution $x^* \in S$ such that $x^* \ne y$.

Figure gives candidate $H = \{x \in I\!R^n : (x^* - y)^T x = (x^* - y)^T x^*. \}$

Take $x \in S$ and $\alpha \in (0, 1)$. By the convexity of S and the definition of x^* we obtain

$$\|(1 - \alpha)x^* + \alpha x - y\|_2^2 \ge \|x^* - y\|_2^2.$$

Rearrangements and simplifications give

$$\alpha \|x - x^*\|_2^2 + 2(x - x^*)^T (x^* - y) \ge 0,$$

implying that $(x - x^*)^T (x^* - y) \ge 0$.

It now follows that the candidate H has the required properties. □

Farkas' lemma

Lemma. *Let A be an $m \times n$ matrix, and let b be an m-dimensional vector. Then exactly one of the following systems has a solution.*

$$1. \quad Ax = b, \; x \ge 0, \qquad\qquad 2. \quad A^T y \ge 0, \; b^T y < 0.$$

Proof. Both systems cannot have a solution simultaneously. (Try.)

It remains to show that if 1 has no solution, then 2 has a solution.

Let $\mathcal{W} = \{w \in I\!R^m : w = Ax, \; x \ge 0\}$. Then $\mathcal{W}$ is a closed convex set. (Closedness follows by reducing to basic feasible solutions.)

Then 1 has a solution if and only if $b \in \mathcal{W}$.

If $b \notin \mathcal{W}$, then by the separating hyperplane theorem, there is a $y \in I\!R^m$ such that $y^T b < y^T z$ for all $z \in \mathcal{W}$. This y satisfies 2. $\square$

Suggested reading

Suggested reading in the textbook:

- Sections 2.1–2.6.
- Sections 3.1–3.2.