KTH Mathematics

2E5295/5B5749 Convex optimization with engineering applications

Lecture 3

Linear programming and the simplex method

Optimality conditions

For f differentiable, consider

$$(P) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in S \subseteq \mathbb{R}^n. \end{array}$$

Proposition. *Assume that S is a convex subset of $\mathbb{R}^n$, and assume that $f : S \to \mathbb{R}$ is a convex differentiable function on S. Then, $x^* \in S$ is a* global minimizer *to (P) if and only if $\nabla f(x^*)^T(x - x^*) \geq 0$ for all $x \in S$.*

This condition is not immediate to verify, since it involves all feasible x.

We will consider more immediate conditions.

Linear program

A *linear program* is a convex optimization problem on the form

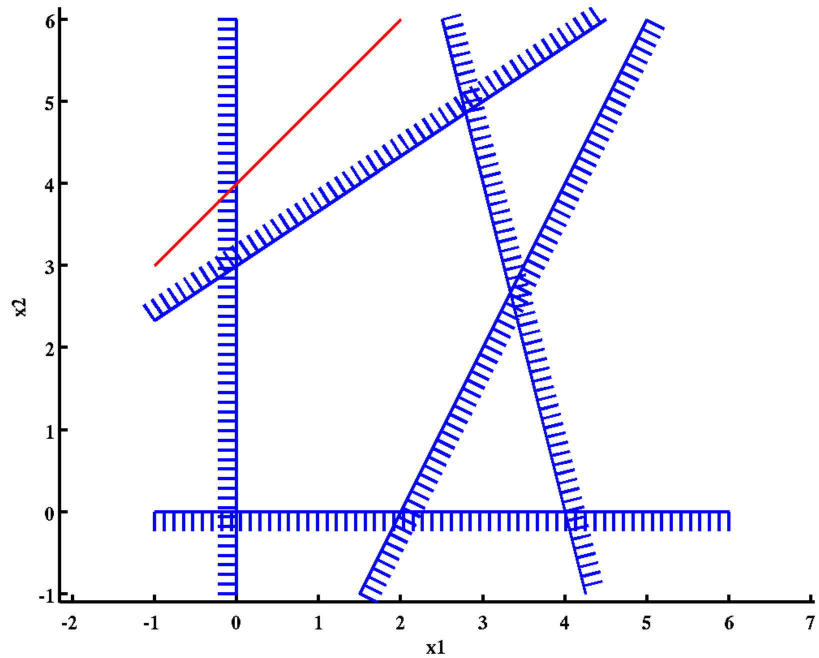
$$(LP) \qquad \begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array}$$

May be written on many (equivalent) forms.

The feasible set is a *polyhedron*, i.e., given by the intersection of a finite number of hyperplanes in $\mathbb{R}^n$.

Example linear program

$$
\begin{aligned}
\text{min} \quad & -x_1 + x_2 \\
\text{subject to} \quad & -2x_1 + x_2 \geq -4, \\
& 2x_1 - 3x_2 \geq -9, \\
& -4x_1 - x_2 \geq -16, \\
& x_1 \geq 0, \\
& x_2 \geq 0.
\end{aligned}
$$



Example linear program, cont.

Equivalent linear programs.

$$\begin{array}{ll} \min & -x_1 + x_2 \\ \text{subject to} & -2x_1 + x_2 \geq -4, \\ & 2x_1 - 3x_2 \geq -9, \\ & -4x_1 - x_2 \geq -16, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{array}$$

$$\begin{array}{ll} \min & -x_1 + x_2 \\ \text{subject to} & -2x_1 + x_2 - x_3 = -4, \\ & 2x_1 - 3x_2 - x_4 = -9, \\ & -4x_1 - x_2 - x_5 = -16, \\ & x_j \geq 0, \quad j = 1, \dots, 5. \end{array}$$

Example linear program, cont.

Equivalent linear programs.

$$
\begin{array}{ll}
\min & -x_1 + x_2 \\
\text{subject to} & -2x_1 + x_2 \geq -4, \\
& 2x_1 - 3x_2 \geq -9, \\
& -4x_1 - x_2 \geq -16, \\
& x_1 \geq 0, \\
& x_2 \geq 0.
\end{array}
$$

$$
\begin{array}{ll}
\min & -x_1 + x_2 \\
\text{subject to} & 2x_1 - x_2 + x_3 = 4, \\
& -4x_1 + 4x_2 - x_3 + x_4 = 5, \\
& 4x_1 - x_2 - x_3 - x_4 + x_5 = 3 \\
& x_j \geq 0, \quad j = 1, \dots, 5.
\end{array}
$$

Methods for linear programming

We will consider two type of methods for linear programming.

- The simplex method.
  - Combinatoric in its nature.
  - The iterates are extreme points of the feasible region.
- Interior methods.
  - Approximately follow a trajectory created by a perturbation of the optimality conditions.
  - The iterates belong to the relative interior of the feasible region.

Standard form and inequality form

We will consider linear programs on standard form,

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array}$$

By partitioning $A = (B\ N)$ where B is $m \times m$ and invertible we obtain

$$\begin{array}{ll} \min & c_B^T x_B + c_N^T x_N \\ \text{subject to} & Bx_B + Nx_N = b, \\ & x_B \geq 0, \quad x_N \geq 0. \end{array}$$

Standard form and inequality form, cont.

We will consider linear programs on standard form,

$$\begin{array}{ll} \text{min} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array}$$

Elimination of x_B as $x_B = B^{-1}b - B^{-1}Nx_N$ gives

$$\begin{array}{ll} \text{min} & (c_N - N^T B^{-T} c_B) x_N \\ \text{subject to} & -B^{-1}Nx_N \geq -B^{-1}b, \\ & x_N \geq 0. \end{array}$$

Equivalent problem on inequality form.

Linear program and extreme points

Definition. *Let S be a convex set. Then x is an* extreme point *to S if $x \in S$ and there are no $y \in C$, $z \in C$, $y \neq x$, $z \neq x$, and $\alpha \in (0, 1)$ such that $x = (1-\alpha)y + \alpha z$.*

$$
(LP) \qquad \begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array}
$$

Theorem. *Assume that (LP) has at least one optimal solution. Then, there is an optimal solution which is an extreme point.*

One way of solving a linear program is to move from extreme point to extreme point, requiring decrease in the objective function value. (The simplex method.)

Linear program extreme points and basic feasible solutions

Proposition. *Let* $S = \{x \in I\!R^n : Ax = b$ *where* $A \in I\!R^{m\times n}$ *of rank* $m\}$. *Then, if* x *is an extreme point of* S, *we may partition* $A = (B\ N)$ *(column permuted), where* B *is* $m \times m$ *and invertible, and* x *conformally, such that*

$$\begin{pmatrix} B & N \\ 0 & I \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad \text{with } x_B \geq 0 \quad .$$

Note that $x_B = B^{-1}b$, $x_N = 0$.

We refer to B as a *basis matrix*.

Extreme points are referred to as *basic feasible solutions*.

Optimality of basic feasible solution

Assume that we have a basic feasible solution

$$\begin{pmatrix} B & N \\ 0 & I \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Proposition. *The basic feasible solution is optimal if* $c^T p^i \geq 0$*,* $i = 1, \ldots, n-m$*, where* p^i *is given by*

$$\begin{pmatrix} B & N \\ 0 & I \end{pmatrix} \begin{pmatrix} p_B^i \\ p_N^i \end{pmatrix} = \begin{pmatrix} 0 \\ e_i \end{pmatrix}, \quad i = 1, \ldots, n-m.$$

Proof. If $\widetilde{x}$ is feasible, it must hold that $\widetilde{x} - x = \sum_{i=1}^{n-m} \gamma_i p^i$, where $\gamma_i \geq 0$, $i = 1, \ldots, n-m$. Hence, $c^T(\widetilde{x} - x) \geq 0$. $\square$

Test of optimality of basic feasible solution

Note that $c^T p^i$ may be written as

$$c^T p^i = \begin{pmatrix} c_B^T & c_N^T \end{pmatrix} \begin{pmatrix} B & N \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ e_i \end{pmatrix}.$$

Let y and s_N solve $\begin{pmatrix} B^T & 0 \\ N^T & I \end{pmatrix} \begin{pmatrix} y \\ s_N \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix}.$

Then $c^T p^i = \begin{pmatrix} y^T & s_N^T \end{pmatrix} \begin{pmatrix} 0 \\ e_i \end{pmatrix} = (s_N)_i.$

We may compute $c^T p^i$, $i = 1, \ldots, n - m$, by solving *one* system of equations.

An iteration in the simplex method

- Compute simplex mulipliers y and reduced costs s from

$$\begin{pmatrix} B^T & 0 \\ N^T & I \end{pmatrix} \begin{pmatrix} y \\ s_N \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix}.$$

- If $(s_N)_t < 0$, compute search direction p from

$$\begin{pmatrix} B & N \\ 0 & I \end{pmatrix} \begin{pmatrix} p_B \\ p_N \end{pmatrix} = \begin{pmatrix} 0 \\ e_t \end{pmatrix}.$$

- Compute maximum steplength $\alpha_{\max}$ and limiting constraint r from

$$\alpha_{\max} = \min_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}, \quad r = \operatorname*{argmin}_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}.$$

- Let $x = x + \alpha_{\max} p$.

- Replace $(x_N)_t = 0$ by $(x_B)_r = 0$ among the active constraints.

An iteration in the simplex method, alternatively

- Compute simplex mulipliers y and reduced costs s from

$$B^T y = c_B, \quad s_N = c_N - N^T y.$$

- If $(s_N)_t < 0$, compute search direction p from

$$p_N = e_t, \quad B p_B = -N_t.$$

- Compute maximum steplength α_{max} and limiting constraint r from

$$\alpha_{\text{max}} = \min_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}, \quad r = \operatorname*{argmin}_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}.$$

- Let $x = x + \alpha_{\text{max}} p$.

- Replace $(x_N)_t = 0$ by $(x_B)_r = 0$ among the active constraints.

To find a basic feasible solution

A basic feasible solution can be found by solving a *Phase I-problem*

$$\begin{array}{ll} \text{min} & e^T u \\ \text{subject to} & Ax + u = b, \\ & x \geq 0, \quad u \geq 0, \end{array}$$

where e has all components 1.

We assume that $b \geq 0$. (If not, the correpsponding rows in $Ax = b$ may be multiplied by -1.)

Basic feasile solution to the Phase I-problem is given with u as basic variables.

Solve the Phase I-problem by the simplex method. If the optimal value is zero, we have a basic feasible solution to the original problem, otherwise the original problem is infeasible.

Primal and dual linear programs

For a *primal* linear program

$$
(PLP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}
$$

we will associate a *dual* linear program

$$
(DLP) \qquad \begin{array}{ll} \text{max} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}
$$

(We will derive the dual by *Lagrangian relaxation* later.)

Weak duality for linear programming

Proposition. *If* x *is feasible to* (PLP) *and* y, s *is feasible to* (DLP), *then* $c^T x - b^T y = x^T s \geq 0$.

Proof. Insertion gives the result. $\square$

It follows that *weak duality* holds, i.e., $\text{optval}(PLP) \geq \text{optval}(DLP)$.

Proposition. *If* x *is feasible to* (PLP) *and* y, s *is feasible to* (DLP), *and furthermore* $x^T s = 0$, *then these solutions are optimal to the primal and the dual, respectively.*

Proof. A consequence of the previous result. $\square$

Note that if $x \geq 0$ and $s \geq 0$, then $x^T s = 0$ if and only if $x_j s_j = 0$, $j = 1, \ldots, n$.

Strong duality for linear programming

$$(PLP) \quad \begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array} \qquad (DLP) \quad \begin{array}{ll} \underset{y \in \mathbb{R}^m}{\text{maximize}} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

Theorem. *If (PLP) has a finite optimal value, then there exist optimal solutions to (PLP) and (DLP). Moreover, the optimal values are equal.*

This result is referred to as *strong duality* for linear programming.