KTH Mathematics

2E5295/5B5749 Convex optimization with engineering applications

Lecture 5

Convex programming and semidefinite programming

Convex quadratic program

A *quadratic program* is an optimization problem on the form

$$(LP) \qquad \begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & \frac{1}{2} x^T H x + c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array}$$

May be written on many (equivalent) forms.

We will only consider *convex* quadratic programs, where $H \succeq 0$.

The nonconvex quadratic programming problem is NP-hard.

Primal and dual quadratic programs

For a *primal* quadratic program

$$(PQP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

we will associate a *dual* quadratic program

$$(DQP) \qquad \begin{array}{ll} \max & -\frac{1}{2}w^T H w + b^T y \\ \text{subject to} & -Hw + A^T y + s = c, \\ & s \geq 0. \end{array}$$

We may derive the dual by Lagrangian relaxation.

If $H \succ 0$, we may eliminate w.

Quadratic programming, optimality conditions

Primal and dual quadratic programs

$$(PQP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax = b, \quad x \geq 0, \end{array}$$

$$(DQP) \quad \begin{array}{ll} \text{max} & -\frac{1}{2}w^T H w + b^T y \\ \text{subject to} & -Hw + A^T y + s = c, \quad s \geq 0. \end{array}$$

Optimality conditions:

$$\begin{aligned} Ax &= b, \\ -Hx + A^T y + s &= c, \\ x_j s_j &= 0, \quad j = 1, \dots, n, \\ x \geq 0, &\quad s \geq 0. \end{aligned}$$

Nonlinearly constrained convex program

Consider a convex optimization problem on the form

$$
(CP) \quad \begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \ge 0, \quad i \in \mathcal{I}, & \mathcal{I} \bigcup \mathcal{E} = \{1,\dots,m\}, \\ & a_i^T x - b_i = 0, \quad i \in \mathcal{E}, & \mathcal{I} \bigcap \mathcal{E} = \emptyset, \\ & x \in I\!\!R^n, & \end{array}
$$

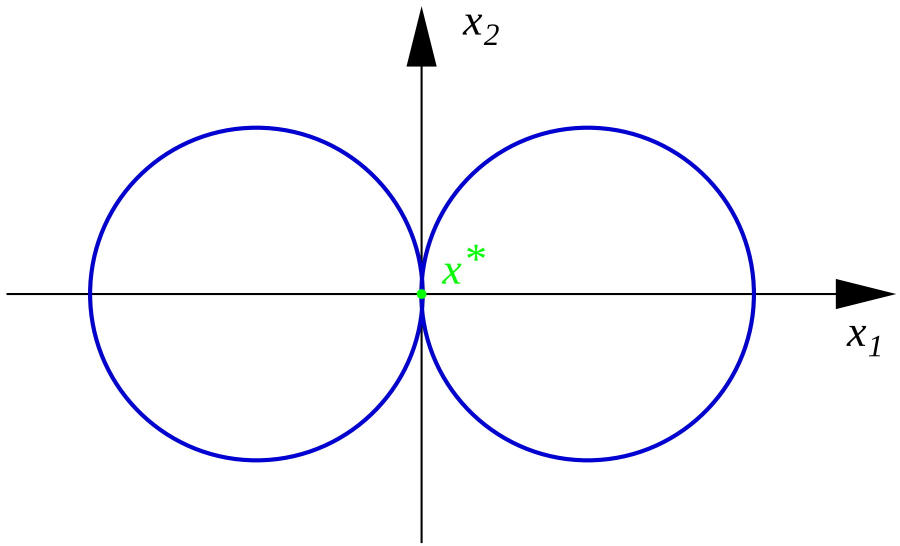
where $f : I\!\!R^n \to I\!\!R$ and $-g_i : I\!\!R^n \to I\!\!R$ are convex and twice continuously differentiable on $I\!\!R^n$.

Let $F = \{x \in I\!\!R^n : g_i(x) \ge 0,\ i \in \mathcal{I}, a_i^T x - b_i = 0,\ i \in \mathcal{E}\}$.

Nonlinearly constrained problems require some regularity

For a nonlinearly constrained problem, the analogous first-order conditions are not always necessary.

Let $g(x) = \begin{pmatrix} -(x_1 - 1)^2 - x_2^2 + 1 \\ -(x_1 + 1)^2 - x_2^2 + 1 \end{pmatrix}$ with $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



The linearization of the constraints at x^* does not describe the feasible region "sufficiently well".

Constraint qualification

If the constraints satisfy some regularity condition, often referred to as a *constraint qualification*, the analogous results hold.

Definition. *Let* $F = \{x \in I\!R^n : g_i(x) \geq 0, \; i \in \mathcal{I}, a_i^T x - b_i = 0, \; i \in \mathcal{E}\}$, *where* $-g_i : I\!R^n \to I\!R$ *are convex and twice continuously differentiable on* $I\!R^n$. *Then, the* Slater constraint qualification *holds if there is a point* $\bar{x}$ *such that* $g_i(\bar{x}) > 0, \; i \in \mathcal{I}, \quad a_i^T \bar{x} - b_i = 0, \; i \in \mathcal{E}$.

With the Slater constraint qualification, strong duality holds.

Optimality conditions for convex program, cont.

Proposition. *Let (CP) be a convex program for which the Slater constraint qualification holds. Then, the point x^* is a minimizer to (CP) if and only if there is a $\lambda^* \in \mathbb{R}^m$ such that*

(i) $g_i(x^*) \geq 0$, $i \in \mathcal{I}$, *and* $a_i^T x^* - b_i = 0$, $i \in \mathcal{E}$,

(ii) $\nabla f(x^*) = \sum_{i\in\mathcal{I}} \nabla g_i(x^*)\lambda_i^* + \sum_{i\in\mathcal{E}} a_i\lambda_i^*$,

(iii) $\lambda_i^* \geq 0$, $i \in \mathcal{I}$,

(iv) $\lambda_i^* g_i(x^*) = 0$, $i \in \mathcal{I}$.

These conditions are often referred to as the *KKT conditions*.

Metric for semidefinite matrices

For x and y in $I\!R^n$, the inner product is given by $x^T y = \sum_{j=1}^{n} x_j y_j$.

The associated norm is the Euclidean norm.

Let $\mathcal{S}^n$ denote the space of symmetric $n \times n$ matrices.

For X and Y in $\mathcal{S}^n$, the inner product is given by

$$\text{trace}(X^T Y) = \text{trace}(XY) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij}.$$

The associated norm is the Frobenius norm.

Proposition. *If* $A \succeq 0$ *and* $B \succeq 0$ *belong to* $\mathcal{S}^n$*, then* $\text{trace}(AB) \geq 0$*.*

Proof. If $A = A^T \succeq 0$, there is an L such that $A = LL^T$. Then $\text{trace}(AB) = \text{trace}(LL^T B) = \text{trace}(L^T BL) \geq 0$. □

Semidefinite programming

In semidefinite programming X is a matrix in $\mathcal{S}^n$. Linear programming is the special case when X is diagonal.

In LP we have constraints $A_i^T x = b_i$, $i = 1, \ldots, m$, where $A_i^T \in I\!R^n$ is row i of the constraint matrix A.

In SDP A_i belongs to $\mathcal{S}^n$ for $i = 1, \ldots, m$, and the constraints become trace$(A_i X) = b_i$, $i = 1, \ldots, m$.

Analogously, LP has the objective function $c^T x$, where $c \in I\!R^n$.

In SDP the objective function becomes trace(CX), where $C \in \mathcal{S}^n$.

A semidefinite program on standard form

A semidefinite program on standard form may be written as

$$(PSDP) \qquad \begin{array}{ll} \text{minimize} & \text{trace}(CX) \\ \text{subject to} & \text{trace}(A_i X) = b_i, \quad i = 1, \dots, m, \\ & X = X^T \succeq 0. \end{array}$$

(If C and A_i, $i = 1, \dots, m$, are diagonal we may choose X diagonal which gives (PLP).)

As for LP the form is not so interesting. The important feature is that we have a linear problem with equality constraints and matrix inequalities.

$(PSDP)$ is a convex problem.

Lagrangian relaxation of a semidefinite program

$$
(PSDP) \qquad \begin{array}{ll} \underset{X \in \mathcal{S}^n}{\text{minimize}} & \text{trace}(CX) \\ \text{subject to} & \text{trace}(A_i X) = b_i, \quad i = 1, \dots, m, \\ & X = X^T \succeq 0. \end{array}
$$

For a given $y \in I\!R^m$, we obtain

$$
\begin{aligned}
\varphi(y) &= \underset{X=X^T \succeq 0}{\text{minimize}} \ \text{trace}(CX) - \sum_{i=1}^{m} y_i(\text{trace}(A_i X) - b_i) \\
&= \begin{cases} b^T y & \text{if } \sum_{i=1}^m A_i y_i \preceq C, \\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

The dual of a semidefinite program

$$(PSDP) \qquad \begin{array}{ll} \underset{X \in \mathcal{S}^n}{\text{minimize}} & \text{trace}(CX) \\ \text{subject to} & \text{trace}(A_i X) = b_i, \quad i = 1, \dots, m, \\ & X = X^T \succeq 0. \end{array}$$

$$(DSDP) \qquad \begin{array}{ll} \underset{y \in I\!R^m}{\text{maximize}} & \sum_{i=1}^m b_i y_i \\ \text{subject to} & \sum_{i=1}^m A_i y_i \preceq C. \end{array}$$

If X is feasible to $(PSDP)$ and y, S are feasible to $(DSDP)$, then

$$\text{trace}(CX) - b^T y = \text{trace}(XS).$$

Semidefinite programs with no duality gap

Some regularity condition has to be enforced in order to ensure that there is no duality gap.

For example, the *Slater condition*, which requires a strictly feasible X, i.e., an X that satisfies

$$\text{trace}(A_i X) = b_i, \quad i = 1, \ldots, m,$$
$$X = X^T \succ 0.$$

Similar requirement on $(DSDP)$ ensures no duality gap and existence of optimal solutions to both $(PSDP)$ and $(DSDP)$.

Semidefinite optimality

The set $\mathcal{W}$ defined by

$$\mathcal{W} = \{w \in I\!R^m : w_i = \text{trace}(A_i X),\ i = 1, \dots, m,\ X = X^T \succeq 0\}$$

is *not* closed.

As an example, let $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

For $\epsilon \geq 0$, let $w(\epsilon) = (2 \ \ \epsilon)^T$. Then, $w(\epsilon) \in \mathcal{W}$ for $\epsilon > 0$, where the associated $X(\epsilon)$ is given by

$$X(\epsilon) = \begin{pmatrix} \epsilon & 1 \\ 1 & x_{22}(\epsilon) \end{pmatrix} \quad \text{for} \quad x_{22}(\epsilon) \geq \frac{1}{\epsilon}.$$

However, $w(0) \notin \mathcal{W}$.

A semidefinite program may not have an optimal solution

Let $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b_1 = 2$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The primal problem has optimal value 0 but no optimal solution.

The dual problem has optimal value 0 and an optimal solution.

(Wolkowicz, Saigal and Vandenberghe: *Handbook of Semidefinite Programming*, Example 4.1.1.)

A semidefinite program may have duality gap

Let $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$C = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where a is a given positive number.

The primal problem has optimal value a and an optimal solution.

The dual problem has optimal value 0 and an optimal solution.

(Wolkowicz, Saigal and Vandenberghe: *Handbook of Semidefinite Programming*, Example 4.1.2.)

Suggested reading

Suggested reading in the textbook:

- Section 4.4.
- Sections 5.4–5.9.