EL3300/SF3849 Convex Optimization with Engineering Applications Autumn 2010
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Homework Assignment 4
Due Wednesday January 5 2010

Exercise 4.1. Let $A = A^T \in \mathbf{R}^{m\times m}$, $B \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{n\times n}$ and

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

Show that $M \succeq 0$ if and only if $A \succeq 0$, $C \succeq 0$ and

$$|u^T Bv|^2 \leq (u^T Au)(v^T Cv)$$

for all $u \in \mathbf{R}^m$ and $v \in \mathbf{R}^n$.

Exercise 4.2. For $W = W^T \succ 0$ let $\mathcal{X} = \{x : x^T Wx \leq 1\}$. Show that

$$(A - BL)x \in \mathcal{X},\ \forall x \in \mathcal{X}$$

if and only if

$$(A - BL)^T W(A - BL) - W \preceq 0. \tag{4.2.1}$$

Exercise 4.3. In this exercise, we are going to derive a distributed optimization scheme for joint end-to-end rate and link-rate adaption of a wireless network. This model extends the TCP model by Low and Lapsley [2] to a wireless setting. Feel free to consult the original paper, but most of the mathematical details that you will need are on the slides for the lecture on large-scale and distributed optimization.

Consider a data network formed by a collection of nodes located at fixed positions in the plane; see Figure 1(left). Each node is assumed to have infinite buffering capacity and can transmit, receive and relay data to other nodes across wireless links. We represent the topology of the network by a directed graph, with nodes labeled $n = 1, \dots, N$ and links labeled $l = 1, \dots, L$. A link is represented by an ordered pair (i, j) of distinct nodes. The presence of link (i, j) means that the network is able to send data from the start node i to the end node j.

The capacity (maximum data rate) on an individual link depends on the medium access scheme and the allocation of radio communication resources to the transmitters of the links. To make things simple, we will assume that the radio channels are orthogonal and that the capacity of each link c_l only depends on the transmit power P_l allocated to the link. In particular, we will assume that

$$c_l(P_l) = W \log(1 + \alpha_l P_l), \qquad l = 1, \dots, L, \tag{4.3.1}$$

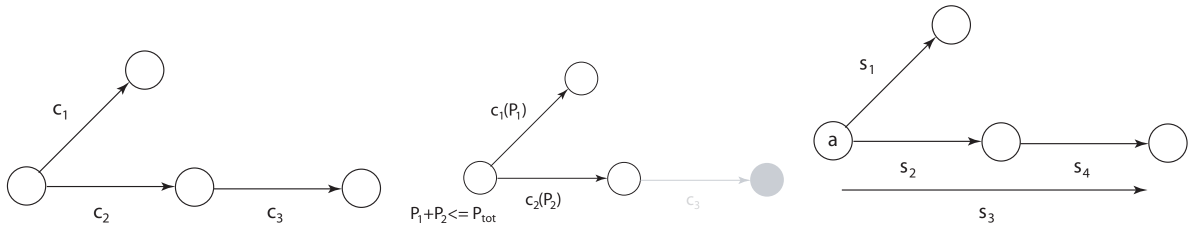

Figure 1: Network with three directed links (left); the link capacities depend on the transmit powers, and the total power is limited for each node (middle); each node can, in principle, communicate with every other node, creating at most $N(N-1)$ node pairs (right). Due to the structure of this particular network, where links are only unidirected, only four source-destination pairs can communicate.

for some fixed strictly positive constants W and α_l. We will further assume that the instantaneous power of each node is limited to P_{tot}. To describe this constraint mathematically, let $\mathcal{O}(n)$ denote the set of outgoing links of node n. Then,

$$\sum_{l\in\mathcal{O}(n)} P_l \leq P_{\text{tot}}, \qquad n = 1,\dots,N. \tag{4.3.2}$$

In this network, there are $P = N(N-1)$ possible node-pairs which may want to exchange data. We label the source-destination pairs by integers $p = 1,\dots,P$ and let s_p denote the data rate communicated between source-destination pair p, and let $s = \begin{pmatrix} s_1 & \cdots & s_P \end{pmatrix}$ be the vector of end-to-end communication rates. Some node pairs may be able to communicate by direct links, while other node pairs need to rely on other nodes for forwarding their traffic from source to destination. To represent the routes, we introduce a routing matrix $R = [r_{lp}] \in \mathbb{R}^{L\times P}$ whose entries

$$r_{lp} = \begin{cases} 1 & \text{if the traffic between node pair } p \text{ traverses link } l, \\ 0 & \text{otherwise.} \end{cases}$$

The vector of total traffic across the links is given by Rs, and the end-to-end rates must satisfy the following constraints

$$Rs \preceq c(P),$$

or, equivalently, $r_l^T s \leq c_l(P_l)$ for $l = 1,\dots,L$, where r_l^T is the lth row of R.

We let $u_p(s_p)$ represent the utility that communication pair p has to exchange data at rate s_p. Note that each node acts as source for $N-1$ source-destination pairs, and that it has separate (and possibly different) utility functions for each connection. The utility functions $u_p(\cdot)$ are assumed to be concave and increasing in their arguments. We now seek the combination of transmit powers and end-to-end rate that maximizes the total network utility, *i.e.*, we want to solve the optimization problem

$$\begin{array}{lll} \text{maximize} & \sum_{p=1}^{P} u_p(s_p) & & \text{(total network utility)} \\ \text{subject to} & r_l^T s \leq c_l(P_l), & l = 1,\dots,L, & \text{(capacity constraints on links)} \\ & \sum_{l\in\mathcal{O}(n)} P_l \leq P_{\text{tot}}, & n = 1,\dots,N. & \text{(total power in nodes)} \end{array}$$

a) Show that this problem is convex, and that strong duality holds (Hint. Construct a simple allocation that establishes that the feasible set has a strict interior point.)

b) Form the partial Lagrangian to the utility maximization problem by relaxing the capacity constraints only.

c) Determine the Lagrange dual function associated with the Lagrangian derived in Exercise 4.3b. Demonstrate that the dual function is separable in end-to-end rates and transmit powers. Furthermore, show that the rate optimization problem can be solved by each source-destination pair separately, and that the power allocation problem can be solved individually by each node. What information do sources/nodes need from the network in order to perform their optimization tasks?

d) The power allocation subproblem will have the structure

$$\begin{array}{ll} \text{maximize} & \sum_{l \in \mathcal{O}(n)} w_l \log(1 + \alpha_l P_l) \\ \text{subject to} & \sum_{l \in \mathcal{O}(n)} P_l \leq P_{\text{tot}}, \qquad P_l \geq 0, \end{array}$$

for some non-negative weights w_l. Use the KKT conditions to derive an efficient procedure for solving the above problem (Hint. See Section 5.5.3 in *Convex optimization.*)

e) Formulate the Lagrange dual problem, and derive the subgradient update step. Show how each link can update its associated multiplier individually based on local information.

References

[1] D. M. Gay. Electronic mail distribution of linear programming test problems. *Math. Prog. Society COAL Newsletter*, 13:10–12, 1985.

[2] S. H. Low and D. E. Lapsley. Optimization flow control — I: basic algorithm and convergence. *IEEE/ACM Transactions on Networking*, 7(6):861–874, 1999.

[3] Netlib linear programming test problems. http://www-fp.mcs.anl.gov/otc/Guide/TestProblems/LPtest/.

[4] S. J. Wright. *Primal-Dual Interior-Point Methods.* SIAM, Society for Industrial and Applied Mathematics, Philadelphia, 1997. ISBN 0-89871-382-X.

Good luck!