

Lecture 11: applications in control

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Disposition

Control applications today, communications on Thursday.

Some insight about two key techniques:

- Finite-horizon control for discrete-time linear systems
 - linear or quadratic programming
 - basic building block in model-predictive control
- Stability and performance of continuous-time linear systems
 - semidefinite programming and Lyapunov theory
 - useful for design of state and output feedback laws

Most of the time will be devoted to the second topic.

Finite-horizon optimal control

Linear system with finite-time control objective

$$\begin{aligned} & \text{minimize} && \sum_{t=1}^T f_t(x(t), u(t); x(0)) \\ & \text{subject to} && x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ & && x(t) \in X_t, \quad u(t) \in U_t \end{aligned}$$

Given initial value, pick control sequence that minimizes cost.

Convex problem if f_t convex function, X_t and U_t convex sets

Examples

Which problems do the following formulations address?

1. minimize $\sum_{t=1}^T u(t)^2$
subject to $x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$
 $X_T = \{x_f\}$
2. minimize $\sum_{t=1}^T x(t)^T Qx(t) + u(t)^T Ru(t)$
subject to $x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$
 $|u(t)| \leq u_{\max}, \quad \|x(t)\|_{\infty} \leq x_{\max}$
3. minimize $\sum_{t=1}^T (x(t) - r(t))^2 + \lambda |u(t) - u(t-1)|$
subject to $x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$

Key technique: variable elimination

Example: finite-time horizon linear-quadratic problem

$$\begin{aligned} & \text{minimize} && \sum_{t=1}^T x(t)^T Q x(t) + u(t)^T R u(t) \\ & \text{subject to} && x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ & && Cx(t) + Du(t) \leq \eta, \quad t = 1, \dots, T \end{aligned}$$

Use the relation

$$x(t) = A^t x_0 + A^{t-1} B u_0 + \dots + A B u_{t-2} + B u_{t-1} = H_t \begin{bmatrix} x_0 \\ u \end{bmatrix}$$

to eliminate states $x(t)$ in problem formulation. Equivalent problem

$$\begin{aligned} & \text{minimize} && \begin{bmatrix} x_0 \\ u \end{bmatrix}^T \begin{bmatrix} Q_e & S_e^T \\ S_e & R_e \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \\ & \text{subject to} && C_e x_0 + D_e u \leq \eta_e \end{aligned}$$

Model predictive control

1. Given state at time t compute (“predict”) future states

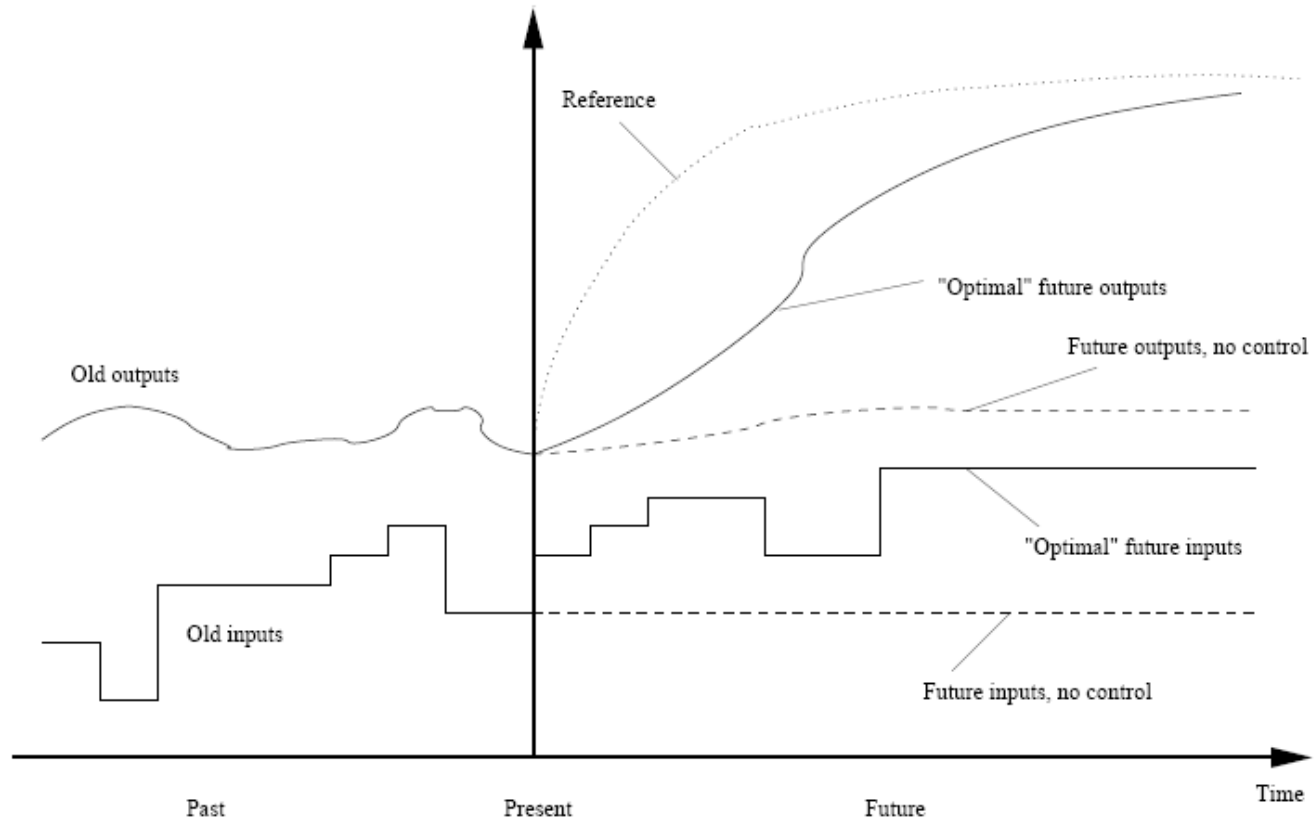
$$x(t + k), \quad k = 1, \dots, T$$

as function of future control inputs

$$u(t + k), \quad k = 1, \dots, T$$

2. Find “optimal” input by minimizing finite-horizon cost
3. Implement $u(t)$
4. A next sample $(t+1)$, return to 1.

Model predictive control in pictures



Explicit MPC

By convex duality (Lecture 5), u is optimal if and only if (u, λ) satisfies

$$\begin{aligned} R_e u + S_e x_0 + D_e^T \lambda &= 0 \\ \lambda^{(i)} (C_e^{(i)} x_0 + D_e^{(i)} u - \eta_e^{(i)}) &= 0 \quad i = 1, \dots, M \\ \lambda &\geq 0 \\ C_e x_0 + D_e u - \eta_e &\leq 0 \end{aligned}$$

Optimal control is locally affine in x_0 ,

$$u = -R_e^{-1} (S_e x_0 + D_e^T \lambda)$$

within a polyhedron defined by the other optimality constraints.

MPC-equivalent feedback is piecewise linear, can be computed off-line!

Content

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Linear matrix inequalities

An inequality on the form

$$F(x) = F_0 + \sum_{i=1}^n x_i F_i > 0$$

where F_i are symmetric matrices, and x_1, \dots, x_n are scalar unknowns

Note: F_i can be complex-valued Hermetian matrices

Some standard problems

SDPs (convex)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) = F_0 + \sum_{i=1}^n x_i F_i > 0 \end{array}$$

Generalized eigenvalue problems (quasi-convex)

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \lambda B(x) - A(x) > 0, \quad B(x) > 0, \quad C(x) > 0 \end{array}$$

Max-det problems (convex)

$$\begin{array}{ll} \text{minimize} & \log \det A(x)^{-1} \\ \text{subject to} & A(x) > 0 \quad B(x) > 0 \end{array}$$

SDP dual

Recall that the dual to the SDP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) = F_0 + \sum_{i=1}^n x_i F_i \succ 0 \end{array}$$

is the SDP

$$\begin{array}{ll} \text{maximize} & \text{Tr}(F_0 Z) \\ \text{subject to} & \text{Tr}(F_i Z) = c_i \\ & Z \succeq 0 \end{array}$$

S-procedure

Does it hold that

$$f(x) \geq 0 \quad \text{for all } x \in C : g(x) \leq 0$$

Equivalently: do we have

$$\inf_{x \in C, g(x) \leq 0} f(x) \geq 0$$

By weak and strong duality, respectively:

- yes, if there exists $\lambda \geq 0$ with

$$f(x) + \lambda^T g(x) \geq 0 \quad \text{for all } x \in C$$

- Suppose C, f, g , convex and $\exists x_0 \in C$ with $g(x_0) < 0$, then above conditions are necessary and sufficient

S-procedure cont'd

Nontrivial result without convexity:

Let $f, g: \mathbb{R}^n \mapsto \mathbb{R}$ be quadratic functions such that $\exists x_0$ with $g(x_0) < 0$, then the answer is yes if and only if there exists $\lambda \geq 0$ such that

$$f(x) + \lambda g(x) \geq 0 \quad \text{for all } x \in C$$

Let $f(x) = x^T F x$ and $g(x) = x^T G x$, then the above condition reads

$$F + \lambda G \geq 0$$

an LMI in the variable λ

Example: quadratic confinement

Let $\mathcal{Q}_i = \{ x \mid q_i(x) = x^T A_i x + 2b_i^T x + c_i \geq 0 \}$. Does $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$?

Yes, if there is $\lambda \geq 0$ such that

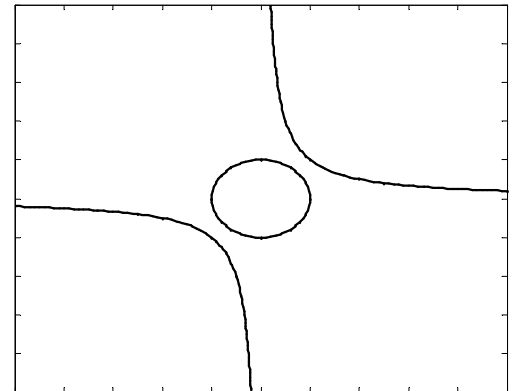
$$\begin{pmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{pmatrix} - \lambda \begin{pmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{pmatrix} \geq 0$$

Example: does: $\{x : -x_1^2 - x_2^2 + 1 \geq 0\} \subseteq \{x : -2x_1 x_2 + 2 \geq 0\}$

Yes, since

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \geq 0$$

for, e.g., $\lambda = 1.5$



One (of many) extensions

We have

$$C + X^T B + B^T X + X^T A X \geq 0 \text{ for all } X \text{ with } I - X^T D X \geq 0$$

If and only if there exists $\lambda \geq 0$ such that

$$\begin{bmatrix} C & B^T \\ B & A \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \\ 0 & -D \end{bmatrix} \geq 0$$

Schur complement

The condition

$$M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$$

holds if and only if

$$Q > 0 \quad R - S^T Q^{-1} S > 0$$

Schur complement proof

The condition

$$M > 0$$

is equivalent to

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^T Q x + 2x^T S y + y^T R y > 0 \quad \text{for all } (x, y)$$

$$\text{i.e. } \inf_x x^T Q x + 2x^T S y + y^T R y > 0 \quad \text{for all } y$$

The minimum is attained for $x = -Q^{-1}S y$, so this is equivalent to

$$y^T [R - S^T Q^{-1} S] y > 0 \quad \text{for all } y$$

Congruence transforms

The condition

$$M > 0$$

holds if and only if

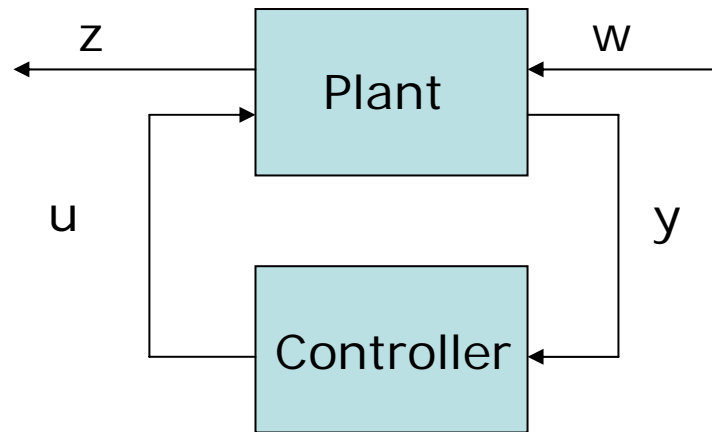
$$T^T M T > 0$$

for some nonsingular matrix T .

Example. Schur complement proved by congruence transform

$$T = \begin{bmatrix} I & -Q^{-1}S \\ 0 & I \end{bmatrix}$$

Control system setup



Design controller that maps measurements y into controls u

- Stability of closed-loop system
- Performance (e.g., suppression of disturbance w)
- Many different control structures; here full state feedback

Stability analysis via Lyapunov functions

The system $\dot{x} = Ax$ is exponentially stable iff there exists P with

$$P > 0 \quad A^T P + PA < 0$$

In particular, there exists $\varepsilon > 0$ such that all trajectories satisfy

$$\|x(t)\| \leq \|x(0)\| \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\varepsilon t}$$

Note: multiple LMIs still LMI, Lyapunov inequalities equivalent to

$$\begin{bmatrix} -P & 0 \\ 0 & A^T P + PA \end{bmatrix} < 0$$

Lyapunov proof

The Lyapunov inequalities imply that there is $\epsilon > 0$ such that

$$A^T P + PA + \epsilon P \leq 0$$

Let $V(x) = x^T P x$. Then, for all $t \in \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt} V(x(t)) + \epsilon V(x(t)) &= x^T(t) [A^T P + PA] x(t) + \epsilon x^T P x \\ &= x^T [A^T P + PA + \epsilon P] x \leq 0 \end{aligned}$$

After integration, this yields, for all $t \geq t_0$

$$x^T(t) P x(t) \leq x^T(t_0) P x(t_0) e^{-\epsilon t}$$

The result now follows from

$$\lambda_{\min}(P) \|x\|^2 \leq x^T P x \leq \lambda_{\max}(P) \|x\|^2$$

Simultaneous Lyapunov function

Consider the piecewise linear system

$$\dot{x} = A_i x \quad \text{for } x \in X_i$$

All trajectories of the system go to zero exponentially if there is P :

$$\begin{aligned} P &> 0 \\ A_i^T P + P A_i &< 0, \quad \forall i \end{aligned}$$

Still LMI conditions.

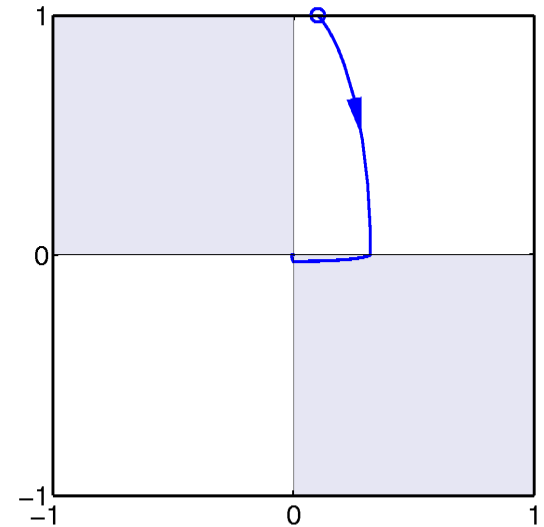
Challenge: derive less conservative conditions when

$$X_i = \{x \mid x^T Q_i x \geq 0\}$$

Example:

Consider the piecewise linear system

$$\dot{x} = \begin{cases} \begin{bmatrix} -0.1 & 1 \\ -10 & -0.1 \end{bmatrix} & \text{if } x_1 x_2 \geq 0 \\ \begin{bmatrix} -0.1 & 10 \\ -1 & -0.1 \end{bmatrix} & \text{if } x_1 x_2 \leq 0 \end{cases}$$



No common Lyapunov function unless partition information exploited!

Simultaneous stabilization

Given the system

$$\dot{x} = A_i x + B_i u \quad \text{for } x \in X_i$$

Is there a state feedback $u=Lx$, which stabilizes the system?

$$\dot{x} = A_i x + B_i u \quad \text{for } x \in X_i$$

Lyapunov conditions read

$$P > 0 \quad (A_i + B_i L)^T P + P(A_i + B_i L) < 0$$

Not LMIs! Considering congruence transform with $Q=P^{-1}$ gives

$$Q > 0 \quad Q A_i^T + A_i Q + Q L^T B_i^T + B_i L Q < 0$$

LMI conditions in (Q, Y) where $Y = LQ$

No common Lyapunov function

For given A_i , how would you prove that there is no solution P to

$$\begin{aligned} P &> 0 \\ A_i^T P + P A_i &< 0, \quad \forall i \end{aligned}$$

Claim: if there exist $R_i > 0$ such that

$$\sum_i R_i A_i^T + A_i R_i > 0$$

then there is no P satisfying the Lyapunov inequalities above.

Proof:

Example: L_2 gain analysis

Closed loop linear system with disturbance w and output y

$$\dot{x} = Ax + Bw \quad y = Cx \quad x(0) = 0$$

has L_2 gain γ if, for all (square integrable) w and all $t \geq 0$,

$$\int_{\tau=0}^t y(\tau)^T y(\tau) d\tau \leq \gamma^2 \int_{\tau=0}^t w(\tau)^T w(\tau) d\tau$$

Alternative characterization: non-negative storage function $V(x) \geq 0$

$$\frac{\partial V}{\partial x}(Ax + Bw) \leq \gamma^2 w^T w - y^T y$$

How to find storage function? How to find the best one (smallest γ)?

L₂ gain analysis via convex optimization

Fact: for linear systems, sufficient to consider quadratic $V(x)$

$$V(x) = x^T P x$$

Gain conditions translate to *linear matrix inequalities* (in P, γ^2)

$$\begin{array}{c} P > 0 \\ \left[\begin{array}{cc} A^T P + P A + C^T C & P B \\ B^T P & -\gamma^2 I \end{array} \right] < 0 \end{array}$$

Convex conditions: can find optimal P and smallest γ efficiently!

State feedback design

Find state feedback law $u = Lx$ that minimizes L_2 induced gain $w \rightarrow y$

$$\dot{x} = Ax + B_u u + Bw$$

$$y = Cx$$

Claim: If there exists Q , γ and Y such that

$$\begin{bmatrix} AQ + QA^T + B_u Y + Y^T B_u^T + \gamma^{-2} B B^T & Q C^T \\ C Q & -I \end{bmatrix} < 0 \quad Q > 0$$

then, the state feedback $u = YQ^{-1}x$ makes the induced L_2 -gain of the closed-loop system less than γ

Proof:

Much more

The framework has many extensions:

- H_2 and H_∞ designs
- Constraints on closed loop pole locations
- Dynamic output feedback
- Systems in discrete-time
- Uncertain systems
- Nonlinear and parameter-varying systems
- Hybrid systems
- \vdots