IMPORTANT MOMENTS IN SYSTEMS AND CONTROL*

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Dedicated to the memory of Mark Grigoryevich Krein on the occasion of the 100th anniversary of his birth

Abstract. The moment problem matured from its various special forms in the late 19th and early 20th Centuries to a general class of problems that continues to exert profound influence on the development of analysis and its applications to a wide variety of fields. In particular, the theory of systems and control is no exception, where the applications have historically been to circuit theory, optimal control, robust control, signal processing, spectral estimation, stochastic realization theory and the use of the moments of a probability density. Many of these applications are also still works in progress. In this paper, we consider the generalized moment problem, expressed in terms of a basis of a finite-dimensional subspace $\mathfrak{P}$ of the Banach space $C[a,b]$ and a "positive" sequences c, but with a new wrinkle inspired by the applications to systems and control. We seek to parameterize solutions which are positive "rational" measures, in a suitably generalized sense. Our parameterization is given in terms of smooth objects. In particular, the desired solution space arises naturally as a manifold which can be shown to be diffeomorphic to a Euclidean space and which is the domain of some canonically defined functions. The analysis of these functions, and related maps, yields interesting corollaries for the moment problems and its applications, which we compare to those in the recent literature and which play a crucial role in part of our proof. Our techniques are a combination of those drawn from the literature on the generalized moment problem, from the topology of smooth manifolds and maps, and from convex optimization.

Key words. Moment problem

AMS subject classifications. 30E05, 44A60, 93B28

1. Introduction. Given a sequence of complex numbers, $(c_0, c_1, \cdots, c_n)$, and a basis, $(\alpha_0, \alpha_1, \cdots, \alpha_n)$, of a (finite-dimensional) subspace $\mathfrak{P}$ of the Banach space $C[a,b]$ of complex-valued continuous functions defined on the real interval $[a,b]$, the generalized moment problem [21] is to find a positive measure $d\mu$ such that

$$\int_a^b \alpha_k(t)d\mu(t) = c_k, \quad k = 0, 1, \cdots, n. \tag{1.1}$$

This problem is a beautiful generalization of several important classical moment problems, including the power moment problem, the trigonometric moment problem and the moment problem arising in Nevanlinna-Pick interpolation. There are, of course, necessary conditions stemming from the positivity of $d\mu$ and whether a particular α_k is real-valued or not; these will be summarized in Section 2.

Among the pioneers in the use of power moments, where $\alpha_k(t) = t^k$, we should mention Chebyschev and his students, particularly Markov and Lyapunov, who used them in connection with the classical Central Limit Theorem in the 19th Century. On a finite interval this problem is usually called the Hausdorff moment problem and was solved by Hausdorff for an infinite sequence of moments in 1921. The power moment problem for an infinite sequence of moments on an infinite interval is known as the Hamburger moment problem, while on the semi-infinite interval this is called

*This research was supported in part by grants from AFOSR, Swedish Research Council, and the Göran Gustafsson Foundation

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the Stieltjes moment problem. We refer to [21], especially pages 166–171 and the references therein, for a more detailed historical and technical treatment.

Remark 1.1. In classical treatments of the power moment problems [21] it is typical to take $\mathfrak{P}$ to be the real subspace $\operatorname{span}_{\mathbb{R}}\{\alpha_0, \cdots, \alpha_n\}$. While in this case the role of the functions α_k are clear and familiar to any student of probability, it is reasonable to ask why we need $\mathfrak{P}$. One of many good reasons for this is that $\mathfrak{P}$ is a natural space of "test functions" with which to develop necessary and sufficient conditions on the candidate moments for the solvability of the moment equations. For example, if $p(t) = p_0 + p_1 t + \cdots + p_n t^n > 0$ for all $t \in [a, b]$, then solvability of the moment equations for a positive measure $d\mu$ implies that

$$\sum_{i=0}^{n} p_i c_i = \int_a^b p(t) d\mu > 0. \tag{1.2}$$

This has been refined, in a neat way, to give necessary and sufficient conditions for the solvability of the generalized moment problem (see [21] and the discussion in Section 2).

In the trigonometric moment problem, where $\alpha_k(t) = e^{ikt}$ defined on $[-\pi, \pi]$, the constants c_k are, of course, the first $n+1$ Fourier coefficients of $d\mu$. The corresponding moment problem was classically considered by Carathéodory in potential theory, where the moment conditions place a constraint on the boundary value data for Laplace's equation on the unit disc. Through subsequent classical work by Schur, Toeplitz, Nevanlinna, Pick and many others, this has been influential in the development of modern analysis (see e.g. [15]). Applications of the trigonometric moment problem to systems and control also have a long and fruitful history, including the rational covariance extension problem originally posed by Kalman [17] and later observed to be related to the trigonometric moment problem in [11]. However, to be applicable to problems in spectral estimation and stochastic realization theory there are systems theoretic constraints that must be added to the trigonometric moment problem, relating to rationality of, and the degree of, a solution. These challenges were noticed early on [17, 18, 12] and the ultimate breakthroughs relied (and still do rely) on the nontrivial use of topology, nonlinear convex optimization or a combination thereof (see [12, 2], and the SIGEST paper [3] and references therein).

Remark 1.2. In this setting, the classical theory was developed for a complex subspace $\mathfrak{P}$ of "test functions" and follows, *mutatis mutandis*, the real case. Explicitly, in order to develop the corresponding necessary conditions it is necessary to take those $p \in \mathfrak{P}$ for which the trigonometric polynomial $P := \operatorname{Re}(p)$ is positive on $[-\pi, \pi]$. The comples-valued enhacement of condition (1.2) is then

$$\operatorname{Re}\left(\sum_{i=0}^{n} p_i c_i\right) = \frac{1}{2}\sum_{i=0}^{n}(\bar{p}_i c_i + p_i \bar{c}_i) = \int_{-\pi}^{\pi} P d\mu > 0. \tag{1.3}$$

One of the many gems in this classical literature is the use [21, p. 65] of the Riesz-Fejér Theorem to evaluate the quadratic form on the right hand side of (1.3), where $P > 0$, as

$$\sum_{i,j=0}^{n} c_{i-j} z_i \bar{z}_j = \bar{z}^{\mathsf{T}} T_n z > 0,$$

where $z = (z_0, \dots, z_n) \in \mathbb{C}^n \setminus 0$ and T_n the standard Toeplitz form fashioned out of the moment sequence $c = (c_i)$. For a general moment problem, the form on the left hand side of (1.3) is classically denoted by $\langle c, p \rangle$.

Remark 1.3. In both the power and the trigonometric moment problems we were led to consider the "polynomials," $P = \mathrm{Re}(p)$, for $p \in \mathfrak{P}$. For this reason, the functions $P := \mathrm{Re}(p)$, for $p \in \mathfrak{P}$ in an arbitrary generalized moment problem are referred to as "polynomials" for $\mathfrak{P}$. Following this precedent, we shall refer to the ratio P/Q with $p, q \in \mathfrak{P}$ as a "rational functions" for $\mathfrak{P}$.

In the Nevanlinna-Pick moment problem for distinct interpolation points $z_0, z_1, \dots, z_n$, the basis functions are given by

$$\alpha_k(t) = \frac{1}{2\pi} \frac{e^{ikt} + z_k}{e^{ikt} - z_k}, \quad k = 0, 1, \cdots, n,$$

which coincide on $[-\pi, \pi]$, modulo an additive constant, with Cauchy kernels. Higher order kernels can of course be used for multiple points. As for the case of trigonometric polynomials, it turns out that it is more helpful to identify the interval with the unit circle and, in this case, to think of $\mathfrak{P}$ in terms of Hardy spaces. This has also led to profound developments in several complex variables and in operator theory as well as in the applications of mathematics to circuit theory [10, 16] and to robust control [24, 19, 13, 9, 20]. For this problem as well, the applications to systems and control impose additional constraints to the classical moment problem whose treatment still requires nonlinear methods drawn from geometry, topology and/or optimization [16, 13, 5, 3].

Remark 1.4. For the classical Nevanlinna-Pick interpolation problem, using the Riesz-Fejér Theorem, the quadratic form (1.3) can also be evaluated, with some work [21, pp. 67-69] as the value of the celebrated Pick form. Moreover, it turns out that $\mathfrak{P}$ is a finite-dimensional coinvariant subspace of H^2 so that the elements of $\mathfrak{P}$ are rational functions σ/τ, where τ is fixed, and the "polynomials" are the real parts of elements in $\mathfrak{P}$. This of course implies that the "rational functions" are rational in the usual sense.

The generalized moment problem is about measures and combining these two concepts leads us to following definition.

Definition 1.5. Any measure of the form

$$d\mu = \frac{P(t)}{Q(t)} dt, \tag{1.4}$$

where P and Q are positive polynomials for $\mathfrak{P}$, is a (generalized) *rational positive measure*.

Problem 1.6. Given a sequence of complex numbers $c_0, c_1, \cdots, c_n$ and a subspace $\mathfrak{P}$, the generalized moment problem for rational measures is to parameterize all positive rational measures $\frac{P(t)}{Q(t)} dt$ such that

$$\int_a^b \alpha_k(t) \frac{P(t)}{Q(t)} dt = c_k, \quad k = 0, 1, \cdots, n. \tag{1.5}$$

The problem itself is motivated by classical applications and examples, in both finite and infinite dimensions, and also reflects the importance of rational functions in systems and control. In this paper we give a concise description of all solutions of this generalized moment problem for a broad class of subspaces $\mathfrak{P}$.

2. The Main result. In order to state our result, we first need to compute the dimension of $\mathfrak{P}$ as a real vector space, taking into account the cases where a basis element is real, purely imaginary or neither. In order for the moment equations to hold it is necessary that c_k be real whenever α_k is real. Moreover, a purely imaginary moment condition can always be reduced to a real one, and henceforth we shall assume that $\alpha_0, \dots, \alpha_{r-1}$ are real functions and $\alpha_r, \dots, \alpha_n$ are complex-valued functions whose real and imaginary parts, taking together with $\alpha_0, \dots, \alpha_{r-1}$, are linearly independent over $\mathbb{R}$. In particular, we may regard $\mathfrak{P}$ as a real vector space of dimension is $2n - r + 2$. Since we have chosen a fixed basis, we may regard each

$$p := \sum_{k=0}^{n} p_k \alpha_k \in \mathfrak{P} \tag{2.1}$$

also as $(n+1)$-tuple of points $(p_0, p_1, \dots, p_n)$, where $p_0, p_1, \dots, p_{r-1}$ are real and $p_r, p_{r+1}, \dots, p_n$ are complex. Moreover, p is determined by its real part $P := \mathrm{Re}(p)$, a notation we shall keep throughout. Next we define the subset $\mathfrak{P}_+$ of those elements $p \in \mathfrak{P}$ such that $P > 0$. We shall assume that $\mathfrak{P}_+$ is nonempty and is therefore an open, convex set having dimension $2n - r + 2$.

The rational measures we seek as solutions have the property that $d\mu(E) > 0$ for every Borel measurable subset $E \subset [a, b]$ having nonzero measure. For this reason, we will seek a necessary condition expressible in terms of the slightly larger space $\overline{\mathfrak{P}}_+ \smallsetminus \{0\}$ of "test" functions. That is, we define $\mathfrak{C}_+$ as the set of sequences $c = (c_0, c_1, \cdots, c_n)$ such that

$$\langle c, p \rangle := \mathrm{Re}\left\{ \sum_{k=0}^{n} p_k c_k \right\} = \int_a^b P \, d\mu > 0 \tag{2.2}$$

for all $p \in \overline{\mathfrak{P}}_+ \smallsetminus \{0\}$. We will call such a sequence *positive*. In particular, $\mathfrak{C}_+$ is also a nonempty, open convex subset of $\mathbb{R}^{2n-r+2}$ of dimension $2n - r + 2$.

Remark 2.1. Since we are seeking a solution to the moment problem for a smaller class of positive measure than in the classical treatment, our necessary conditions are, not surprisingly, stronger than the classical conditions. In particular, a sequence c which we call positive is referred to as *strictly positive* in [21]. A sequence c is positive *in the classical sense* if it satisfies $\langle c, p \rangle \geq 0$ for all $p \in \overline{\mathfrak{P}}_+$, a condition that does not quite capture the case of positive rational measures.

We shall now fix $c \in \mathfrak{C}_+$ and consider the set $\mathfrak{M}_c$ of all pairs of polynomials (p, q) for which the rational measure $P(t)/Q(t)dt$ solves Problem 1.6 for the positive sequence c. There is a natural parameterization of $\mathfrak{M}_c$ as a submanifold of the product space $\mathfrak{P}_+ \times \mathfrak{P}_+$ and, as a subset of a product space, $\mathfrak{M}_c$ comes with two mappings:

$$\pi_1 : \mathfrak{M}_c \to \mathfrak{P}_+ \quad \text{and} \quad \pi_2 : \mathfrak{M}_c \to \mathfrak{P}_+,$$

where π_1 and π_2 are the restrictions to $\mathfrak{M}_c$ of the two mappings

$$\mathrm{proj}_1 : \mathfrak{P}_+ \times \mathfrak{P}_+ \to \mathfrak{P}_+ \quad \text{and} \quad \mathrm{proj}_2 : \mathfrak{P}_+ \times \mathfrak{P}_+ \to \mathfrak{P}_+,$$

defined by $\mathrm{proj}_1(p, q) = p$ and $\mathrm{proj}_2(p, q) = q$.

Theorem 2.2. *Suppose that $\mathfrak{P}$ consists of Lipschitz continuous functions. Then, for each $c \in \mathfrak{C}_+$, $\mathfrak{M}_c$ is a smooth submanifold of $\mathfrak{P}_+ \times \mathfrak{P}_+$, diffeomorphic to $\mathbb{R}^{2n-r+2}$. Moreover, each of the maps π_1, π_2 is a diffeomorphism of $\mathfrak{M}_c$ onto its image, which is an open submanifold of $\mathfrak{P}_+$. Finally, $\pi_1 : \mathfrak{M}_c \to \mathfrak{P}_+$ is surjective.*

Remark 2.3. To the best of our knowledge, all instances of the generalized moment problem that arise in systems and control involve subspaces of $C[a,b]$ consisting of Lipschitz continuous functions. Moreover, this class of subspaces have been of considerable classical interest. For example, an important class of spaces considered in the classical literature on the generalized moment problem [21] consists of those spaces $\mathfrak{P}$ spanned by a Chebyshev system (or T-system), which are characterized by a bound on the number of zeros for any nonzero polynomial in $\mathfrak{P}$. These spaces arise in important applications of the generalized moment problem; e.g., the power moment problem and the trigonometric moment problem of odd order and have remarkable approximation properties in the Banach space $C[a,b]$. We remark that [21] contains a neat application, generalizing Feldbaum's Theorem on the number of switchings, of Chebyschev systems to the time-optimal control of scalar-input linear control systems. For our present purposes, we recall the classical result that, if $\mathfrak{P}$ is spanned by a Chebyshev system and contains a constant function, then, after a reparameterization, $\mathfrak{P}$ consists of Lipschitz continuous functions [21, p. 37].

Remark 2.4. We have remarked that the finite-dimensional Nevanlinna-Pick problem can be recast in a Hardy space setting, where the space $\mathfrak{P}$ is a coinvariant subspace (defined, in fact, by a finite Blaschke product) in $H^2(\mathbb{D})$. In a seminal paper [22], Sarason developed a vast generalization of this problem to one involving liftings of a partial isometry T, defined on an arbitrary coinvariant subspace, which commute there with the restriction of the shift operator. Among many other results, Sarason showed that, under general conditions, the lift of T has an H^∞ symbol which is *rational* with respect to the coinvariant subspace. The corresponding problem for T being a strict contraction was studied in [8], where optimization methods were used to show that the lifting of such a T always had such a generalized rational symbol. Moreover, it was shown that this symbol is completely parameterized by its numerator in parallel with the conclusion in Theorem 2.2 that π_1 is a bijection. In this light, it is interesting to enquire whether a general version of Problem 1.6 can be formulated, and solved, in a meaningful infinite dimensional setting.

The formulations of Definition 1.5 and Problem 1.6 for generalized rational measures and of Theorem 2.2 are new and have some appeal both for the intrinsic simplicity of the formulation and as a unification of a variety of specific applications and more general results on the moment problem. There are of course antecedents in the literature to some parts of the theorem and its corollaries. We shall review these results as a conclusion to our outline of the proof in Section 3.

3. An outline of the proof. The proof of our main result can be reduced to several steps. The first part involves establishing some smoothness results for $\mathfrak{M}_c$ and the maps π_1 and π_2. This, of course, depends upon the ambient spaces and their properties, as investigated in Section 4. In Proposition 4.1, we establish the required smoothness and prove that each of the maps π_1 and π_2 is a local diffeomorphism, whenever $\mathfrak{M}_c$ is nonempty.

The final steps in the proof are to demonstrate that $\mathfrak{M}_c$ is nonempty for each positive sequence c, that π_1 is a bijection and that π_2 is an injection. For, suppose that $\mathfrak{M}_c$ is nonempty. By the Inverse Function Theorem, the image of each π_i is an open subset U_i of $\mathfrak{P}_+$. Therefore, to say that π_1 is also a bijection, is to say that it has an inverse defined on $U_1 = \mathfrak{P}_+$, which from the Inverse Function Theorem must also be differentiable. That is, the map

$$\pi_1 : \mathfrak{M}_c \to \mathfrak{P}_+$$

is a diffeomorphism. Similarly, to say that π_2 is an injection is to say that

$$\pi_2 : \mathfrak{M}_c \to U_2$$

is a diffeomorphism. Taken together, these steps conclude the proof. The proofs of the last three steps are, however, not just set-theoretic.

For example, the analysis of the map π_2 boils down to the analysis of a linear map between closed convex sets. If $\mathfrak{P}$ contains the constant functions, we may, for example, choose $q = 1$ which leads to a new constrained problem, the generalized moment problem for positive *polynomial measures*. In Section 5, after proving in Lemma 5.1 that π_2 is injective, we analyze its image using the auxiliary problem for polynomial measures. In particular, we deduce Proposition 5.3 which asserts that π_2 fails to be surjective for general c in dimension greater than one.

In contrast, the analysis of π_1 is nonlinear and the result is nicer. To say that for every c and for each p, there exists a unique q is to say that for each fixed p and any c, there exists a unique q so that the corresponding rational measure solves the moment problem for c. As for the map π_2, this results in a related constrained moment problem, but in this case there turns out to be a "Dirichlet principle." Briefly, such a principle should assert (in analogy with the inverse problem of the calculus of variations) that these moment equations (in analogy with the Euler-Lagrange equations) should represent the critical point equations for some variational criterion. Moreover, in the best of all possible worlds, the corresponding critical points would turn out to be minimizing and unique. In Section 6 we show that this, indeed, occurs for the moment problem for this class of rational measures. In particular, we note in Proposition 6.2 that the map π_1 is injective.

Injectivity of π_1 is equivalent to the uniqueness of solutions to the related generalized moment problem introduced in Section 6. The existence of solutions to this moment problem follows from *a priori* bounds for the solutions in terms of bounds on the moment data. In Section 7, the existence of these a priori bounds are established in Lemma 7.6, which holds whenever $\mathfrak{P}$ consists of Lipschitz continuous functions. This has several important and interesting corollaries. It allows us to prove Theorem 7.1 asserting the smooth dependence of these solutions on initial data. This yields Corollary 7.2 which asserts that $\mathfrak{M}_c$ is nonempty, for each positive sequence c. Finally, we deduce (see Proposition 7.3) that π_1 is surjective and hence a bijection, thereby concluding the proof of Theorem 2.2.

Remark 3.1. The proof of Theorem 2.2 both touches upon and gives new proofs of certain results in the literature on generalized moment problems with a degree, or a complexity, constraint. Some of these were developed in some specific applications to problems arising in systems and control and, later, in a more general setting. For example, in the SIGEST paper [3], we surveyed the trigonometric moment problem and its manifestation in our work, and the work of Georgiou, on the covariance extension problem. In [3], we also reviewed our joint work with Georgiou [5] on the Nevanlinna-Pick moment problem. In both of these problems, a specialized version of Theorem 7.1 emerged. It is fair to say that, at the time, everybody interested in this circle of problems recognized that this kind of result capped off the brilliant introduction of topological methods into these problems by Georgiou [12, 13]. Motivated by the similarities between these problems, between their corresponding solutions and their common role as classical instances of the generalized moment problem, we concluded [3] with a sketch of a unified approach to both applications in the form of a constrained generalized moment problem, as treated in Sections 6 and 7. The resulting formulation stated a version of Theorem 7.1 for arbitrary subspaces $\mathfrak{P}$ and referred, as did the more recent survey [6], to the unpublished report [4] for more details and proofs. However, it is also fair to say that, at the time, both the formulation of the general problem in terms of (generalized) rational measures and of Theorem 2.2 remained unanticipated.

The basic technical lemma in [4] has been generalized here as Lemma 7.6, and is proved in the case when $\mathfrak{P}$ consists of Lipschitz continuous functions. This is unlikely to be the most general form of the technical lemma but, in the light of Remark 2.3, could be the most interesting form for finite dimensional subspaces $\mathfrak{P}$. A brief overview of this result and the hypotheses under which versions of Theorem 7.1 have been established can be described as follows.

- The proof of the corresponding results in [4] required that the subspace $\mathfrak{P}$ consists of functions of class C^2.
- Georgiou [14] developed an innovative approach to the generalized moment problem with complexity constraints based on a one-parameter embedding argument, similar to the path-lifiting proof of the Banach-Mazur Theorem in [1]. Using this method, Georgiou was able to prove an analogue of Theorem 7.1 for subspaces $\mathfrak{P}$ consisting of functions of class C^1.
- In [7], another alternate approach to this constrained moment problem was developed from a detailed analysis of the underlying variational problem (see Remark 7.5), proving in particular that all minimizers arise as interior points. The proof holds under a condition concerning certain divergent integrals that is valid whenever $\mathfrak{P}$ consists of Lipschitz continuous functions.

In contrast, the approach followed here is to use only the existence of the underlying variational criterion, without solving the variational problem, to give a streamlined yet self-contained proof of existence and uniqueness results for a class of constrained moment problems en route to our ultimate goal, Theorem 2.2.

4. Some basic results on smoothness. We now turn to the smoothness of $\mathfrak{M}_c$ and the maps π_1 and π_2. The map

$$M : \mathfrak{P}_+ \times \mathfrak{P}_+ \to \mathfrak{C}_+,$$

defined via

$$M(p,q) = \int_a^b \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} \frac{P(t)}{Q(t)} dt$$

has $\mathfrak{M}_c$ as its level set $M^{-1}(c)$.

For simplicity, we view $\mathfrak{P}$ and $\mathfrak{C}$ as real vector spaces, so that $\mathfrak{P}$ is spanned by the real basis (α_i), where we have replaced a complex-valued (α_k) by its real and imaginary parts. The Jacobian, $\mathrm{Jac}(M)_{(p_0,q_0)}$, of M at a point (p_0, q_0) takes the form

$$\mathrm{Jac}(M) = (\partial M/\partial p, \partial M/\partial q) = (M_p, M_q) \tag{4.1}$$

where M_p is the square matrix whose (i,j)-th entry is

$$(M_p)_{(i,j)} = \int_a^b \alpha_i(t)\alpha_j(t) \frac{1}{Q(t)} dt \tag{4.2}$$

and where M_q is defined by

$$(M_q)_{(i,j)} = -\int_a^b \alpha_i(t)\alpha_j(t)\frac{P(t)}{Q^2(t)}dt, \tag{4.3}$$

each being evaluated at the point (p_0, q_0). Thus, M_p (or $-M_q$) is the gramian matrix of the real basis (α_i) with respect to the positive definite inner product defined by $P(t)dt$ (or $P(t)/Q^2(t)dt$) on $C[a,b]$. Therefore, $\mathrm{Jac}(M)$ has rank $2n-r+2$ at each point (p,q) so that, by the Implicit Function Theorem, we obtain the following result.

PROPOSITION 4.1. *For each $c \in \mathfrak{C}_+$, $\mathfrak{M}_c$ is either empty or a submanifold of $\mathfrak{P}_+ \times \mathfrak{P}_+$ of real dimension $2n-r+2$.*

As restrictions of a smooth map to a smooth submanifold of the product, both π_1 and π_2 are smooth maps from $\mathfrak{M}_c$ to $\mathfrak{P}_+$. Suppose that $M(p_0,q_0) = c$ so that, in particular, $\mathfrak{M}_c$ is nonempty. The tangent space $T_{(p_0,q_0)}(\mathfrak{M}_c)$ to $\mathfrak{M}_c$ at (p_0,q_0) is given by the kernel of $\mathrm{Jac}(M)_{(p_0,q_0)}$. By inspection, we have

$$\ker \mathrm{Jac}(M)_{(p_0,q_0)} = \left\{ \begin{bmatrix} M_p^{-1}x \\ -M_q^{-1}x \end{bmatrix} : \; x \in \mathbb{R}^{2n-r+2} \right\}. \tag{4.4}$$

We wish to show that

$$\operatorname{rank} \mathrm{Jac}(\pi_1)_{(p_0,q_0)} = 2n-r+2.$$

This will occur if, and only if,

$$\dim \ker \mathrm{Jac}(\pi_1)_{(p_0,q_0)} = 0,$$

which, since $\pi_1 = \mathrm{proj}_1|_{\mathfrak{M}_c}$, is equivalent to the condition that the subspace

$$\ker \mathrm{Jac}(\mathrm{proj}_1)_{(p_0,q_0)} \cap \ker \mathrm{Jac}(M)_{(p_0,q_0)} \tag{4.5}$$

is trivial. Now, since

$$\ker \mathrm{Jac}(\mathrm{proj}_1)(p_0,q_0) = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : \; y \in \mathbb{R}^{2n-r+2} \right\},$$

the intersection (4.5) is parametrized by solutions to the equation $M_p^{-1}(x) = 0$. Since this implies $x = 0$, it follows that the intersection (4.5) is the trivial subspace $\{0\}$.

In paticular, the Jacobian of π_1 at (p_0,q_0) is nonsingular. A similar argument shows that Jacobian of π_2 at (p_0,q_0) is nonsingular and therefore, the final result in this section then follows from the Inverse Function Theorem.

PROPOSITION 4.2. *Whenever $\mathfrak{M}_c$ is nonempty, each of the maps π_1 and π_2 is a local diffeomorphism.*

5. Injectivity of π_2 and the generalized moment problem for polynomial measures. For each fixed $q \in \mathfrak{P}_+$, the inverse image $\pi_2^{-1}(q)$ in $\mathfrak{M}_c$ is convex. If $\pi_2^{-1}(q)$ is nonempty, then Proposition 4.2 implies that it consists of a single point.

LEMMA 5.1. *The map π_2 is an injection.*

The question of whether π_2^{-1} is nonempty is more interesting. To this end, we shall now keep q fixed and vary p. It follows from the above that the corresponding map

$$L_+ : \mathfrak{P}_+ \to \mathfrak{C}_+$$

from a positive polynomial p to a positive sequence c is a convex injection. To say that $L_+(p) = c$ is to say that $(p, q) \in \mathfrak{M}_c$, so that we are interested in the image of L_+. This is also of independent interest. For example, if $\mathfrak{P}$ contains the constant functions, choosing $q = 1$ leads to a special case of Problem 1.6.

PROBLEM 5.2. Given a sequence of complex numbers $c_0, c_1, \cdots, c_n$ and a subspace $\mathfrak{P}$, the generalized moment problem for polynomial measures is to parameterize all positive polynomial measures $P(t)dt$ such that

$$\int_a^b \alpha_k(t)P(t)dt = c_k, \quad k = 0, 1, \cdots, n. \tag{5.1}$$

To say Problem 5.2 is solvable for all positive c is to say that L_+ is surjective, which is in turn equivalent to asserting $L_+ : \partial\mathfrak{P}_+ \to \partial\mathfrak{C}_+$. This is trivially true for $\dim(\mathfrak{P}) = 1$.

PROPOSITION 5.3. *The convex injection L_+ fails to be surjective in dimension greater than one.*

Proof. Indeed, it is suffices to consider the real case and to suppose α_k is a real, orthonormal basis for $\mathfrak{P}$. In this case, L_+ is the identity map and $\mathfrak{C}_+$ can be shown to be the positive orthant in $\mathbb{R}^{2n-r+2}$. If L_+ were a bijection, then it would follow that $\alpha_k \in \partial\mathfrak{P}_+$ for each α_k. Consequently, each α_k would be non-negative on $[a, b]$, contradicting orthogonality in dimensions two or greater. $\square$

This shows that the generalized moment problem for polynomial measures is unsolvable for a set of positive sequences having positive measure, whenever $\mathfrak{P}$ has dimension at least two. The same underlying argument used for the polynomial measure problem works for arbitrary q, if we choose (α_k) to be an orthonormal basis with respect to the positive measure $d\mu = dt/Q$.

6. A Dirichlet principle for the existence of certain rational measures. We now turn to the map π_1. As in the previous section, we shall begin with an analysis of the "fiber" $\pi_1^{-1}(p)$ of the map π_1 over a fixed p in $\mathfrak{P}_+$. As before, for an arbitrary $c \in \mathfrak{C}_+$, this leads to a related constrained moment problem, defined as follows.

Consider the function $F^p : \mathfrak{P}_+ \to \mathfrak{C}_+$, defined componentwise via

$$F_k^p(q) = \int_a^b \alpha_k(t)\frac{P(t)}{Q(t)}dt,$$

and a given positive sequence $c = (c_0, ..., c_n)$. In this notation, the generalized moment problem takes the form

$$F_k^p(q) - c_k = 0, \quad k = 0, 1, \ldots, n.$$

In this setting, our Dirichlet Principle amounts to the observation that these equations are the critical point equations for some variational criterion $\mathbb{J}_c$ defined for $q = \sum_{k=0}^n q_k\alpha_k$ in $\mathfrak{P}_+$. To this end, we fashion the 1-form

$$\omega_c = \mathrm{Re}\left\{\sum_{k=0}^n [c_k - F_k(q)]\, dq_k\right\} = \mathrm{Re}\sum_{k=0}^n c_k dq_k - \int_a^b \frac{P}{Q}dQdt,$$

on $\mathfrak{P}_+$. Computing the exterior derivative we obtain

$$d\omega_c = \int_a^b \frac{P}{Q^2}dQ \wedge dQdt = 0,$$

establishing that the 1-form ω_c is closed.

Now, any convex region is star-shaped so that, since $\mathfrak{P}_+$ is open in $\mathbb{R}^{2n-r-2}$, by the Poincaré Lemma [23] there exist a smooth function $\mathbb{J}_c$ obtained by integrating ω_c on any path between any two endpoints. Since $\mathbb{J}_c$ is unique up to a constant of integration, we can express $\mathbb{J}_c$ as a function of the upper limit Q of the integral, i.e.,

$$\mathbb{J}_c(q) = \int \omega_c = \langle c, q \rangle - \int_a^b P \log Q \, dt.$$

We note that $\mathbb{J}_c$ is strictly convex and has an interior critical point precisely at a solution of the generalized moment problem. This has several nontrivial consequences.

First, any such solution corresponds to a minimum of $\mathbb{J}_c$ and is therefore unique. In other words, $F^p : \mathfrak{P}_+ \to \mathfrak{C}_+$ is injective. Second, the Jacobian, $\mathrm{Jac}(F^p)$, must satisfy

$$\mathrm{Jac}(F^p)^T = \mathrm{Jac}(F^p) = D^2(\mathbb{J}_c) > 0.$$

Summarizing these two observations, we see that

$$F^p : \mathfrak{P}_+ \to \mathfrak{C}_+$$

is a diffeomorphism onto its image, which is an open subset of $\mathfrak{C}_+$.

Remark 6.1. Given a fixed choice of p and c, this shows uniqueness of q. As a corollary, we conclude the following basic result.

Proposition 6.2. *The map $\pi_1 : \mathfrak{M}_c \to \mathfrak{P}_+$ is injective.*

7. Bijectivity of π_1 and certain of its consequences. The key remaining ingredient in the proof of Theorem 2.2 relies on showing that F^p, and hence π_1, is surjective. We have already established that F^p is a diffeomorphism onto its image. If F^p were surjective, it would have a global continuous (in fact smooth) inverse so that the inverse image under F^p of a compact set will be compact. In other words, F^p would necessarily be a *proper* map. Conversely, if F^p is proper then the image of F^p, which we know to be open, is also closed. Since $\mathfrak{C}_+$ is convex, it is connected and therefore F^p is onto.

Before establishing this fundamental property in Lemma 7.6, we develop some of its consequences.

Theorem 7.1. *[7] If $\mathfrak{P}$ consists of Lipschitz continuous functions, the mapping*

$$F^p : \mathfrak{P}_+ \to \mathfrak{C}_+$$

is a diffeomorphism.

Corollary 7.2. *If $\mathfrak{P}$ consists of Lipschitz continuous functions then, for each $c \in \mathfrak{C}_+$, the submanifold $\mathfrak{M}_c$ is nonempty.*

Proposition 7.3. *If $\mathfrak{P}$ consists of Lipschitz continuous functions, for each $c \in \mathfrak{C}_+$ the restriction*

$$\pi_1 : \mathfrak{M}_c \to \mathfrak{P}_+$$

of the first projection is bijective. That is, for every positive sequence c and every choice of p in $\mathfrak{P}_+$, there is a unique q such that (p, q) lies in $\mathfrak{M}_c$.

The conclusion of Proposition 7.3 defines, for each fixed $c \in \mathfrak{C}_+$, a map

$$g^c : \mathfrak{P}_+ \to \mathfrak{P}_+$$

where $g^c(p)$ is the unique q such that $(p,q) \in \mathfrak{M}_c$. This map was studied in [7]. In more explicit terms, q is the unique function in $\mathfrak{P}_+$ such that Q is the denominator in the rational measure with numerator P solving the moment equations

$$\int_a^b \alpha_k(t)\frac{P(t)}{Q(t)}dt = c_k, \quad k = 0, 1, \cdots, n$$

for c. Moreover, in the language of Theorem 2.2, we see that

$$g^c = \pi_2 \circ \pi_1^{-1}.$$

We summarize these observations in the following result.

Corollary 7.4. *[7] The mapping*

$$g^c : \mathfrak{P}_+ \to \mathfrak{P}_+$$

is a diffeomorphism onto its image.

Remark 7.5. Theorem 7.1 and Corollary 7.4 were derived in [7] from a detailed analysis of the optimization problem defined by $\mathbb{J}_c$, which showed in particular that F^p is surjective. Briefly, since $\langle c, q\rangle > 0$, a comparison of logarithmic and linear growth in the definition of $\mathbb{J}_c$ implies that $\mathbb{J}_c$ is bounded from below on $\overline{\mathfrak{P}}_+$ for a positive sequence $c_0, c_1, \dots, c_n$. However, to show that $\mathbb{J}_c$ achieves a (unique) minimum on $\overline{\mathfrak{P}}_+$ requires a proof of properness of $\mathbb{J}_c$, a proof that relies on an analysis of certain divergent integrals associated to $\partial\mathfrak{P}_+$. A more refined analysis is required to ultimately prove that $\mathbb{J}_c$ has an interior minimum. The importance of this for the moment problem is clear: To say that $\mathbb{J}_c$ always has an interior minimum is of course to say that the moment equations

$$F^p(q) = c$$

always have a solution.

The next lemma approaches properness from a different perspective.

Lemma 7.6. *Suppose $\mathfrak{P}$ is a vector space consisting of Lipschitz continuous functions. Then, $F^p : \mathfrak{P}_+ \to \mathfrak{C}_+$ is proper.*

Proof. First note that whenever $f := \sum_{k=0}^n f_k\alpha_k \in \mathfrak{P}_+$, the measure

$$dm = \langle c, f\rangle dt$$

is of course a positive measure, absolutely continuous with respect to the Lebesgue measure dt. In particular,

$$\mathrm{Re}\sum_{k=0}^n f_k F_k^p(q) = \int_a^b \frac{P}{Q}\, dm > 0. \tag{7.1}$$

We claim that for any compact set K in $\mathfrak{C}_+$, $(F^p)^{-1}(K)$ is bounded. To see this, suppose (c_j) is a sequence in $\mathfrak{C}_+$ converging to $c \in \mathfrak{C}_+$ such that $F^p(q_j) = c_j$ for some $q_j \in \mathfrak{P}_+$. We claim that the sequence $M_j := \|q_j\|$ is bounded in any norm on the vector space $\mathfrak{P}_+$. Setting $r_j := q_j/M_j$, we first observe

$$F_k^p(q_j) = M_j \int_a^b \alpha_k \frac{P}{R_j}\, dt$$

so that choosing $f = (f_0, f_1, \dots, f_n)$ as above, it follows that

$$M_j \int_a^b \frac{P}{R_j}\, dm = \langle f, c_j \rangle$$

is a convergent sequence with the limit $\langle f, c \rangle > 0$. Since dm is a positive measure and since the sequence (P/R_j) is bounded away from zero, it follows that the sequence (M_j) is bounded. Therefore, the preimage of a convergent sequence in K has a cluster point in the closure of $\mathfrak{P}_+$. Such a cluster point q cannot lie on the boundary of $\mathfrak{P}_+$, for then Q would be a nonnegative function in $\mathfrak{P}$ having a zero $t_0 \in [a, b]$ but for which the integral in (7.1) is finite. Since Q is Lipschitz continuous at t_0, by definition, there exists an $\varepsilon > 0$ and an $M > 0$ such that $Q(t) \le M|t - t_0|$ whenever $|t - t_0| < \varepsilon$ and $t \in [a, b]$. In particular, if $t_0 \in (a, b)$,

$$\int_a^b \frac{P}{Q}\, dm \ge \frac{1}{M} \int_{t_0-\varepsilon}^{t_0+\varepsilon} \frac{P}{|t - t_0|}\, dm = +\infty,$$

contrary to assumption. If $t_0 = a$ or $t_0 = b$, a similar estimate holds. Hence, $q \in \mathfrak{P}_+$, establishing that F^p is proper. □

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