The Moment Problem for Rational Measures: Convexity in the Spirit of Krein

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In memory of Mark Grigoryevich Krein on the occasion of the 100th anniversary of his birth

Abstract. The moment problem as formulated by Krein and Nudel'man is a beautiful generalization of several important classical moment problems, including the power moment problem, the trigonometric moment problem and the moment problem arising in Nevanlinna-Pick interpolation. Motivated by classical applications and examples, in both finite and infinite dimensions, we recently formulated a new version of this problem that we call the moment problem for positive rational measures. The formulation reflects the importance of rational functions in signals, systems and control. While this version of the problem is decidedly nonlinear, the basic tools still rely on convexity. In particular, we present a solution to this problem in terms of a nonlinear convex optimization problem that generalizes the maximum entropy approach used in several classical special cases.

Mathematics Subject Classification (2000). Primary 30E05; Secondary 44A60.

Keywords. Moment problems, interpolation, rational positive measures, convex optimization.

1. Introduction

The moment problem for positive measures is the synthesis, over the course of more than 70 years by Krein and his collaborators (see [1, 15] and references therein), of many important classical problems in pure and applied mathematics. This paper is devoted to the study of a class of moment problems, which we refer to as the moment problem for positive *rational* measures, whose formulation reflects the importance of rational functions in signals, systems and control. This class of problems abstracts the recent work of a number of authors [3, 4, 5, 6, 8, 9, 11, 12,

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This research was supported in part by grants from AFOSR, Swedish Research Council, Swedish Foundation for Strategic Research, and the Göran Gustafsson Foundation.

13] who incorporated various complexity constraints into the refinements of the moment problem for arbitrary positive measures [15].

We refer to this problem as the moment problem for positive *rational* measures. In this paper we develop some basic results for this problem, closely following the approach outlined in [15]. Indeed, in Section 2 we recall the fundamental result on the generalized moment problem as derived in [15] using convex cones in finite dimensional function spaces and properties of positive measures. In Section 3 we derive similar basic results for the moment problem for positive rational measures using a topological proof that mirrors the steps in the convexity proof in [15].

While this version of the problem is decidedly nonlinear, one can still develop an approach based entirely on convexity. In particular, in Section 4 we present a synthesis of our topological approach with a nonlinear convex optimization problem that we discovered in the context of interpolation problems [4, 5] and generalized to the case of moment problems with complexity constraints [6, 8, 9, 13]. In fact, the topological approach developed in Section 3 allows us to significantly streamline our previous proofs concerning the convex functional and its extema. Naturally, the optimization problem itself generalizes the maximum entropy approach. Indeed, for cases where the space of test functions lie in the Hardy space on the unit circle, we provide in Section 5 a succinct closed form for the maximum entropy solution. In Section 6 we describe some amplifications of our basic results using differentiable maps and manifolds, a methodology upon which we based an alternative approach to this problem in [8, 9] and which is also streamlined by our topological arguments.

2. The moment problem following Krein and Nudel'man

The fundamental result on the generalized moment problem derived in [15] is based on two results, one about properties of convex cones in finite dimensional spaces of continuous functions and the other about properties of postive measures.

The first result concerns a subspace $\mathfrak{P}$ of the Banach space $C[a, b]$ of complex-valued continuous functions defined on the real interval $[a, b]$ and a choice of basis $(u_0, u_1, \cdots, u_n)$ of $\mathfrak{P}$. If $p \in \mathfrak{P}$ we denote by P its real part $P := \mathrm{Re}(p)$. Following [15], we define the subset $\mathfrak{P}_+$ of those elements $p \in \mathfrak{P}$ such that $P \geq 0$. The space $\mathfrak{P}_+ \subset \mathfrak{P}$ is a closed, convex cone. In terms of the basis (u_i), every $\phi \in \mathfrak{P}^*$ corresponds to a complex sequence $c = (c_0, c_1, \cdots, c_n) \in \mathbb{C}^{n+1}$. Since every ϕ is determined by its real part as a linear functional on $\mathfrak{P}$ as a real vector space, we can characterize elements of the dual cone $\mathfrak{P}_+^{\top}$ as those sequences c satisfying

$$\langle c, p\rangle := \mathrm{Re}\left\{\sum_{k=0}^{n} p_k c_k\right\} \geq 0 \tag{2.1}$$

for all $p \in \mathfrak{P}_+$. Such a sequence is classically called *positive*, and the space of positive sequences is denoted by $\mathfrak{C}_+$. In particular, $\mathfrak{C}_+$ is a closed, convex cone with $\mathfrak{C}_+^{\top} = \mathfrak{P}_+$.

Following [15], consider the curve

$$U(t) = \begin{pmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \quad a \le t \le b.$$

We then define the subset $U = \{U(t) : t \in [a, b]\} \subset \mathbb{C}^{n+1}$, and let $K(U)$ denote its convex conic hull. Clearly, $K(U)^{\mathsf{T}} = \mathfrak{P}_+$, from which follows:

Theorem 2.1 **(**[15]**).** $K(U) = \mathfrak{C}_+$.

We now turn to some results concerning positive measures. Given $c \in \mathbb{C}^{n+1}$, the generalized moment problem (see [15]) is to find a positive measure $d\mu$ such that

$$\int_a^b u_k(t) d\mu(t) = c_k, \quad k = 0, 1, \cdots, n. \tag{2.2}$$

For the sake of brevity, from now on we shall refer to this problem as simply the moment problem, omitting the adjective "generalized". More generally, let

$$\mathfrak{M} : C[a, b]^* \to \mathbb{C}^{n+1} \tag{2.3}$$

be the continuous mapping defined via (2.2) for an arbitrary bounded measure $d\mu \in C[a, b]^*$ and consider the subset $\mathcal{M}_+ \subset C[a, b]^*$ of positive measures.

Lemma 2.2 **(**[15]**).** $\mathfrak{M}(\mathcal{M}_+) \subset \mathfrak{C}_+$.

Proof. If $p \in \mathfrak{P}_+ = \mathfrak{C}_+^{\mathsf{T}}$, then

$$\langle c, p \rangle := \operatorname{Re} \left\{ \sum_{k=0}^{n} p_k c_k \right\} = \int_a^b P \, d\mu \ge 0, \tag{2.4}$$

so that $c \in \mathfrak{C}_+$. □

By Theorem 2.1 and Lemma 2.2, we have that $\mathfrak{M}(\mathcal{M}_+)$ is a convex subset of $K(U)$. On the other hand, by choosing $d\mu = \delta_{t_0}$ for each $t_0 \in [a, b]$, it follows that $U \subset \mathfrak{M}(\mathcal{M}_+)$. In particular, to say that $\mathfrak{M}(\mathcal{M}_+)$ is closed is to say that $K(U) \subset \mathfrak{M}(\mathcal{M}_+)$.

In [15], the Helly Selection Theorem is used to show that $\mathfrak{M}(\mathcal{M}_+)$ is closed in $\mathfrak{C}_+$ under the following hypothesis.

Hypothesis 2.3. There exists $p \in \mathfrak{P}_+$ such that $P > 0$ on $[a, b]$.

Theorem 2.4 **(**[15]**).** *Whenever Hypothesis* 2.3 *holds,* $K(U) = \mathfrak{M}(\mathcal{M}_+)$. *In particular,*

$$\mathfrak{C}_+ = \mathfrak{M}(\mathcal{M}_+). \tag{2.5}$$

Of course, in order for the moment equations to hold it is necessary that c_k be real whenever u_k is real. Moreover, a purely imaginary moment condition can always be reduced to a real one.

Convention 2.5. Henceforth we shall assume that $u_0, \ldots, u_{r-1}$ are real functions and $u_r, \ldots, u_n$ are complex-valued functions whose real and imaginary parts, taking together with $u_0, \ldots, u_{r-1}$, are linearly independent over $\mathbb{R}$.

In particular, we may regard $\mathfrak{P}$ as the real vector space $\mathbb{R}^{2n-r+2}$ and $\mathfrak{C}_+ \subset \mathbb{R}^{2n-r+2}$. Therefore, it follows [15] that $\mathfrak{C}_+$ is a closed convex cone of dimension $2n-r+2$, with interior $\mathring{\mathfrak{C}}_+$ consisting of *strictly positive* sequences c; i.e., those sequences c satisfying

$$\langle c, p \rangle := \operatorname{Re}\left\{ \sum_{k=0}^{n} p_k c_k \right\} > 0 \tag{2.6}$$

for all $p \in \mathfrak{P}_+ \setminus \{0\}$. Assuming Hypothesis 2.3, it then follows that $\mathfrak{P}_+$ is also a closed convex cone of dimension $2n-r+2$, with a nonempty interior $\mathring{\mathfrak{P}}_+$ consisting of those $p \in \mathfrak{P}_+$ for which $\operatorname{Re}(p) > 0$.

3. The Main Results

In the power and the trigonometric moment problems, the elements of the subspace $\mathfrak{P}$ are polynomials and trigonometric polynomials, respectively. In part for this reason, the elements of the subspace $\mathfrak{P}$ in an arbitrary moment problem are referred to as "polynomials in $\mathfrak{P}$". Following this precedent, we shall refer to the ratio p/q with $p, q \in \mathfrak{P}$ as a "rational function". For the classical Nevanlinna-Pick interpolation problem, it turns out that $\mathfrak{P}$ is a coinvariant subspace of H^2 so that the "polynomials" are rational functions σ/τ, where τ is fixed. This of course implies that the rational functions in $\mathfrak{P}$ are rational in the usual sense.

Definition 3.1. The functions $P := \operatorname{Re}(p)$, for $p \in \mathfrak{P}$ in the moment problem are referred to as *real polynomials for* $\mathfrak{P}$. We shall refer to the ratio P/Q with $p, q \in \mathfrak{P}$ as a *real rational functions for* $\mathfrak{P}$.

Remark 3.2. Under Convention 2.5,

$$p := \sum_{k=0}^{n} p_k u_k \in \mathfrak{P} \tag{3.1}$$

corresponds to an $(n+1)$-tuple of points $(p_0.p_1, \ldots, p_n)$, where $p_0, p_1, \ldots, p_{r-1}$ are real and $p_r, p_{r+1}, \ldots, p_n$ are complex. Moreover, p is determined by P [8, p. 165].

The moment problem is about measures and combining these two concepts leads us to following definition.

Definition 3.3. Any measure of the form

$$d\mu = \frac{P(t)}{Q(t)} dt, \tag{3.2}$$

where P, Q are positive real polynomials for $\mathfrak{P}$, is a *rational positive measure.* Let $\mathcal{R}_+ \subset \mathcal{M}_+$ denote the subset of rational positive measures.

Problem 3.4. Given a sequence of complex numbers $c_0, c_1, \cdots, c_n$ and a subspace $\mathfrak{P} = \text{span}(u_0, \dots u_n) \subset C[a,b]$, the *moment problem for rational measures* is to parameterize all positive rational measures $\frac{P(t)}{Q(t)}dt$ such that

$$\int_a^b u_k(t)\frac{P(t)}{Q(t)}dt = c_k, \quad k = 0, 1, \cdots, n. \tag{3.3}$$

We shall need an additional hypothesis to accomodate the restriction to rational positive measures.

Hypothesis 3.5. The space $\mathfrak{P}$ consists of Lipschitz continuous functions.

Remark 3.6. To the best of our knowledge, all instances of the generalized moment problem that arise in systems and control involve subspaces of $C[a,b]$ consisting of Lipschitz continuous functions. Moreover, we recall the classical result that, if $\mathfrak{P}$ is spanned by a Chebyshev system (or T-system) and contains a constant function, then after a reparameterization $\mathfrak{P}$ consists of Lipschitz continuous functions [15, p. 37].

In the setting of Section 2, our first result is the following.

Theorem 3.7. *If Hypotheses* 2.3 *and* 3.5 *hold, then*

$$\mathfrak{M}(\mathcal{R}_+) = \mathring{\mathfrak{C}}_+.$$

In other words, the moment problem for rational measures is solvable if, and only if, the sequence c is strictly positive.

For any $d\mu \in \mathcal{R}_+$, consider the sequence c defined by (2.2) and any $p = \sum_{k=0}^n p_k u_k \in \mathfrak{P}_+ \setminus \{0\}$. Then

$$\langle c, p \rangle := \text{Re}\left\{\sum_{k=0}^n p_k c_k\right\} = \int_a^b P(t)d\mu > 0, \tag{3.4}$$

so that $c \in \mathring{\mathfrak{C}}_+$. This observation yields the rational analogue of Lemma 2.2.

Lemma 3.8. *If Hypothesis* 2.3 *holds, then* $\mathfrak{M}(\mathcal{R}_+) \subset \mathring{\mathfrak{C}}_+$.

The following result implies the reverse inclusion.

Theorem 3.9. *If Hypotheses* 2.3 *and* 3.5 *hold, then* $\mathfrak{M}(\mathcal{R}_+)$ *contains a set which is both open and closed in the convex set* $\mathring{\mathfrak{C}}_+$.

Remark 3.10. The assertions in Theorem 3.9 are the topological analogue, for the case of rational measures, of the convexity assertions used in the proof of Theorem 2.4 for the generalized moment problem, where it was shown that the convex subset $\mathfrak{M}(\mathcal{M}_+) \subset \mathfrak{C}_+ = K(U)$ both contains U and is closed. In light of Lemma 3.8, a point mass δ_{t_0} cannot be realized on $\mathfrak{P}$ by a positive rational measure so that $U \not\subset \mathfrak{M}(\mathcal{R}_+)$. Nonetheless, there exists $\mathcal{P}_+ \subset \mathcal{R}_+$ such that $\mathfrak{M}(\mathcal{P}_+)$ is both open and closed in $\mathring{\mathfrak{C}}_+$.

Indeed, for a fixed $P \in \mathring{\mathfrak{P}}_+$ consider the set

$$\mathcal{P}_+ = \{d\mu \in \mathcal{R}_+ : d\mu = \frac{P}{Q}dt, \ \ Q \in \mathring{\mathfrak{P}}_+\} \tag{3.5}$$

and the restriction of the moment mapping $\mathfrak{M}|_{\mathcal{P}_+} : \mathcal{P}_+ \to \mathring{\mathfrak{C}}_+$.

Proposition 3.11. *If Hypothesis* 2.3 *holds, then* $\mathfrak{M}(\mathcal{P}_+) \subset \mathring{\mathfrak{C}}_+$ *is open.*

Proof. For simplicity, we view $\mathfrak{P}$ and $\mathfrak{C}$ as real vector spaces, so that $\mathfrak{P}$ is spanned by the real basis (u_i), where we have replaced a complex-valued (u_k) by its real and imaginary parts. We shall also parameterize $d\mu \in \mathcal{P}_+$ by $q \in \mathring{\mathfrak{P}}_+$. The Jacobian, $\text{Jac}(\mathfrak{M}|_{\mathcal{P}_+})_{q_0}$, of $\mathfrak{M}|_{\mathcal{P}_+}$ at a point $q_0 \in \mathring{\mathfrak{P}}_+$ is a square matrix M_q whose (i,j)-th entry is

$$(M_q)_{(i,j)} = -\int_a^b u_i(t)u_j(t)\frac{P(t)}{Q^2(t)}dt \tag{3.6}$$

evaluated at the point q_0. Thus, $-M_q$ is the gramian matrix of the real basis (u_i) with respect to the positive definite inner product defined by $P(t)/Q^2(t)dt$ on $C[a,b]$. Therefore, $\text{Jac}(\mathfrak{M}|_{\mathcal{P}_+})_q$ has rank $2n-r+2$ at each point $q \in \mathring{\mathfrak{P}}_+$ so that, by the Implicit Function Theorem, $\mathfrak{M}(\mathcal{P}_+)$ is open. □

Proposition 3.12. *If Hypotheses* 2.3 *and* 3.5 *hold, then* $\mathfrak{M}(\mathcal{P}_+) \subset \mathring{\mathfrak{C}}_+$ *is closed.*

Proof. Suppose

$$\mathfrak{M}(\frac{P}{Q_j}dt) = c^j \in \mathring{\mathfrak{C}}_+ \tag{3.7}$$

and $\lim_{j\to\infty} c^j = c_0 \in \mathring{\mathfrak{C}}_+$. We claim that there exists an $M > 0$ such that $\|Q_j\|_\infty \le M$. To see this note that

$$\lim_{j\to\infty} \int_a^b \frac{P^2(t)}{Q_j(t)}dt = \lim_{j\to\infty}\langle c^j, p\rangle = \langle c, p\rangle > 0. \tag{3.8}$$

Setting $\widetilde{Q}_j = Q_j/\|Q_j\|_\infty$, we have

$$\lim_{j\to\infty}\|Q_j\|_\infty \int_a^b \frac{P^2(t)}{\widetilde{Q}_j(t)}dt = \langle c, p\rangle > 0. \tag{3.9}$$

Since $\int_a^b P^2(t)/\widetilde{Q}_j(t)\,dt \ge \epsilon$ for some $\epsilon > 0$, we must have $\|Q_j\|_\infty \le M < \infty$ for some $M > 0$. Therefore, there is a convergent subsequence in $\mathfrak{P} \cap C[a,b]$,

$$\lim_{k\to\infty} q_k = q_0, \ \ \text{with } q_0 \in \mathfrak{P}_+ \tag{3.10}$$

for which

$$0 < \int_a^b \frac{P^2(t)}{Q_0(t)}dt = \langle c, p\rangle < \infty$$

We claim that $q_0 \in \mathring{\mathfrak{P}}_+$.

Suppose, on the contrary, that $Q_0(t_0) = 0$ for some $t_0 \in [a,b]$, Then, since Q_0 is Lipschitz continuous at t_0, there exists an $\varepsilon > 0$ and an $L > 0$ such that $Q_0(t) \le L|t - t_0|$ whenever $|t - t_0| < \varepsilon$ and $t \in [a,b]$. In particular, if $t_0 \in (a,b)$,

$$\int_a^b \frac{P^2}{Q_0}\,dt \ge \frac{1}{L}\int_{t_0-\varepsilon}^{t_0+\varepsilon} \frac{P^2}{|t-t_0|}\,dt = +\infty,$$

contrary to assumption. If $t_0 = a$ or $t_0 = b$, a similar estimate holds. Hence, $q_0 \in \mathring{\mathfrak{P}}_+$, as claimed. □

Corollary 3.13. *If Hypotheses* 2.3 *and* 3.5 *hold, the moment mapping* $\mathfrak{M}|_{\mathcal{P}_+} : \mathcal{P}_+ \to \mathring{\mathfrak{C}}_+$ *is surjective.*

4. A Dirichlet Principle for the moment problem with rational positive measures

In the course of proving Theorem 3.7 we showed that, for any $c \in \mathring{\mathfrak{C}}_+$ and any choice of $P \in \mathring{\mathfrak{P}}_+$, the moment problem for rational positive measures always has a solution in the set

$$\mathcal{P}_+ = \{d\mu \in \mathcal{R}_+ : d\mu = \frac{P}{Q}dt, \;\; Q \in \mathring{\mathfrak{P}}_+\}. \tag{4.1}$$

In this section, using a convex optimization argument, we show that the surjection $\mathfrak{M}|_{\mathcal{P}_+}$ is injective, and we characterize the unique rational measure as the solution of a variational problem. In fact, we derive both a primal optimization problem and its dual. Remarkably, the moment problem for rational positive measures *is* the set of critical point equations for the dual variational problem. In this classical sense, a nonlinear convex optimization provides an illustration of the Dirchlet Principle for this class of moment problems.

Let $\mathbb{I}_p : C_+[a,b] \to \mathbb{R} \cup \{-\infty\}$ be the relative entropy functional

$$\mathbb{I}_p(\Phi) = \int_a^b P(t)\log\Phi(t)dt, \tag{4.2}$$

which is a generalization of the entropy functional obtained by setting $P = 1$. From Jensen's inequality we see that $\mathbb{I}_p(\Phi) \le \log\left(\int_a^b P\Phi dt\right) \le \int_a^b P\Phi dt < \infty$.

Theorem 4.1. *Assume that Hypotheses* 2.3 *and* 3.5 *hold, and let* $c \in \mathring{\mathfrak{C}}_+$. *Then, for any* $P \in \mathring{\mathfrak{P}}_+$, *the constrained optimization problem to maximize* (4.2) *over* $C_+[a,b]$ *subject to the moment constraints*

$$\int_a^b u_k(t)\Phi(t)dt = c_k, \quad k = 0,1,\dots,n, \tag{4.3}$$

has a unique solution, and it has the form

$$\Phi = \frac{P}{Q}, \quad Q := \mathrm{Re}\{q\}, \tag{4.4}$$

where $q \in \overset{\circ}{\mathfrak{P}}_+$.

The optimization problem of Theorem 4.1, to which we shall refer as the *primal problem*, can be solved by Lagrange relaxation. In fact, we have the Lagragian

$$L(\Phi, q) = \mathbb{I}(\Phi) + \text{Re} \sum_{k=0}^{n} q_k \left[c_k - \int_a^b u_k \Phi dt \right],$$

where $(q_0, q_1, \ldots, q_n) \in \mathbb{R}^r \times \mathbb{C}^{n-r+1}$ are Lagrange multipliers. Then,

$$L(\Phi, q) = \int_a^b P \log \Phi \, dt + \langle c, q \rangle - \int_a^b Q\Phi dt,$$

where $Q = \text{Re}\{q\}$ with $q := \sum_{k=0}^{n} q_k u_k \in \mathfrak{P}$. Clearly, comparing linear and logarithmic growth, we see that the dual functional

$$\psi(q) = \sup_{\Phi \in C_+[a,b]} L(\Phi, q)$$

takes finite values only if $q \in \mathfrak{P}_+$, so we may restrict our attention to such Lagrange multipliers. For any $q \in \mathfrak{P}_+$ and any $\Phi \in C_+[a,b]$ such that P/Φ is integrable, the directional derivative

$$d_{(\Phi,q)} L(h) = \int_a^b \left[\frac{P}{\Phi} - Q \right] h \, dt = 0$$

for all $h \in C[a,b]$ if and only if $\Phi = \frac{P}{Q} \in C_+[a,b]$, which inserted into the dual functional yields

$$\psi(q) = \mathbb{J}_p(q) + \int_a^b P(\log P - 1) dt, \tag{4.5}$$

where $\mathbb{J}_p : \mathfrak{P}_+ \to \mathbb{R} \cup \{\infty\}$ is the strictly convex functional

$$\mathbb{J}_p(q) = \langle c, q \rangle - \int_a^b P \log Q dt. \tag{4.6}$$

As the last term in (4.5) is constant, the dual problem to minimize $\psi(q)$ over $\mathfrak{P}_+$ is equivalent to the convex optimization problem

$$\min_{q \in \mathfrak{P}_+} \mathbb{J}(q). \tag{4.7}$$

Since

$$\frac{\partial \mathbb{J}_p}{\partial q_k} = c_k - \int_a^b u_k \frac{P}{Q} dt, \quad k = 0, 1, \ldots, n,$$

it follows from Corollary 3.13 that the optimization problem (4.7) has an optimal solution $\hat{q} \in \overset{\circ}{\mathfrak{P}}_+$ satisfying the moment equations (3.3). Moreover, since the functional (4.6) is strictly convex, this optimum is unique.

Consequently,

$$\hat{\Phi} := \frac{P}{\hat{Q}} \in C_+[a,b] \tag{4.8}$$

is the unique optimal solution of the primal problem. To see this, observe that $\Phi \mapsto L(\Phi, \hat{q})$ is strictly concave and that $dL_{(\hat{\Phi},\hat{q})}(h) = 0$ for all $h \in C_+[a,b]$. Therefore,

$$L(\Phi, \hat{q}) \leq L(\hat{\Phi}, \hat{q}), \quad \text{for all } \Phi \in C_+[a,b] \tag{4.9}$$

with equality if and only if $\Phi = \hat{\Phi}$. However, $L(\Phi, \hat{q}) = \mathbb{I}_p(\Phi)$ for all Φ satisfying the moment conditions (4.3). In particular, since (4.3) holds with $\Phi = \hat{\Phi}$, $L(\hat{\Phi}, \hat{q}) = \mathbb{I}_p(\hat{\Phi})$. Consequently, (4.9) implies that $\mathbb{I}_p(\Phi) \leq \mathbb{I}_p(\hat{\Phi})$ for all $\Phi \in C_+[a,b]$ satisfying the moment conditions, with equality if and only if $\Phi = \hat{\Phi}$. Hence, $\mathbb{I}_p$ has a unique maximum in the space of all $\Phi \in C_+[a,b]$ satisfying the constraints (4.3), and it is given by (4.8).

This concludes the proof of Theorem 4.1, but we have also proven the following theorem.

Theorem 4.2. *Assume that Hypotheses* 2.3 *and* 3.5 *hold. Let* $(c,p) \in \mathring{\mathfrak{C}}_+ \times \mathring{\mathfrak{P}}_+$*, and set* $P := \mathrm{Re}\{p\}$*. Then the functional* (4.6) *has a unique minimizer* $\hat{q} \in \mathring{\mathfrak{P}}_+$*, and* $\hat{Q} := \mathrm{Re}\{\hat{q}\}$ *is the unique solution to the moment equations*

$$\int_a^b u_k \frac{P}{Q} dt = c_k, \quad k = 0, 1, \dots, n. \tag{4.10}$$

Corollary 4.3. *If Hypotheses* 2.3 *and* 3.5 *hold, the moment mapping* $\mathfrak{M}|_{\mathcal{P}_+} : \mathcal{P}_+ \to \mathring{\mathfrak{C}}_+$ *is a bijection.*

5. Moment problems in a Hardy space setting

Some important special cases of the moment problem is when

$$u_k(t) = g_k(e^{it}) \quad \text{where } g_k \in H^2(\mathbb{D}), \quad k = 0, 1, \dots, n, \tag{5.1}$$

and $[a,b] = [-\pi, \pi]$. A case in point is the trigonometric moment problem when $g_k(z) = \frac{1}{2\pi} z^k$; another is Nevanlinna-Pick interpolaton when $g_k(z) = \frac{1}{2\pi} \frac{z+z_k}{z-z_k}$, where $z_0, z_1, \dots, z_n$ are the (distinct) interpolation points. In both of these cases, $g := (g_0, g_1, \dots, g_n)^\mathsf{T}$ can be represented as

$$g(z) = (I - zA)^{-1} B, \tag{5.2}$$

where A is a $n \times n$ stability matrix and B an n-vector such that (A,B) is a reachable pair; i.e.,

$$G = \int_{-\pi}^{\pi} g(e^{it}) g(e^{it})^* dt > 0. \tag{5.3}$$

Indeed, positive definiteness of G follows readily from the fact that the basis functions are linearly independent. This condition also insures that there is a unique function of the form

$$w(z) = \sum_{k=0}^{n} w_k g_k(e^{it})^*$$

that satisfies

$$\int_{-\pi}^{\pi} g_k(e^{it})w(e^{it})dt = c_k, \quad k = 0, 1, \dots, n,$$

namely the one provided by the unique solution of the system of linear equations

$$\sum_{k=0}^{n} G_{kj} w_j = c_k, \quad k = 0, 1, \dots, n.$$

Consequently, for any $q \in \mathfrak{P}$,

$$\langle c, q \rangle = \int_{-\pi}^{\pi} Q(t)w(e^{it})dt. \tag{5.4}$$

It can be shown that $g_0, g_1, \dots, g_n$ span the coinvariant subspace $\mathcal{K} := H^2 \ominus \phi H^2$, where ϕ is the inner function

$$\phi(z) = \frac{\det(zI - A^*)}{\det(I - zA)}.$$

In view of (5.1), $\mathcal{K}$ is a Hardy space model of $\mathfrak{P}$. Moreover, for any $\psi \in \mathcal{K}$, there is a $v \in \mathcal{K}$ such that $\Psi := \mathrm{Re}\{\psi\} = vv^*$ [7, Proposition 9]. Therefore, for any $q \in \mathfrak{P}_+$, there is an $\mathbf{a} \in \mathbb{C}^n$ such that $Q(t) = a(e^{it})^* a(e^{it})$ where $a(z) := g(z)^*\mathbf{a}$. Then, by (5.4),

$$\langle c, q \rangle = \mathbf{a}^* \int_{-\pi}^{\pi} w(e^{it})g(e^{it})g(e^{it})^* dt \, \mathbf{a} = \mathbf{a}^*\mathbf{P}\mathbf{a}, \tag{5.5}$$

where

$$\mathbf{P} := \frac{1}{2}\int_{-\pi}^{\pi} g(e^{it})[w(e^{it}) + w(e^{it})^*]g(e^{it})^* dt. \tag{5.6}$$

Consequently, $c \in \mathfrak{C}_+$ if and only if $\mathbf{P} \geq 0$, and $c \in \mathring{\mathfrak{C}}_+$ if and only if $\mathbf{P} > 0$. In the trigonometric moment problem $\mathbf{P}$ is the Toeplitz matrix, and in the Nevanlinna-Pick case $\mathbf{P}$ is the the Pick matrix.

If $\mathfrak{P}$ contains constants, then we may determine the maximum-entropy solution, corresponding to setting $P = 1$ in (4.2), in closed form.

Proposition 5.1. *Suppose that the basis functions in* $\mathfrak{P}$ *satisfy* (5.1) *and* $\mathfrak{P}$ *contains constants. Then the maximum-entropy solution is*

$$\hat{\Phi}(t) = \frac{g(0)^*\mathbf{P}^{-1}g(0)}{|g(e^{it})^*\mathbf{P}^{-1}g(0)|^2}, \tag{5.7}$$

where $\mathbf{P}$ *is given by* (5.6).

Proof. We proceed as in [14, 16]. Since, by Jensen's formula [2, p.184], the last term in the dual functional (4.6) (with $P = 1$) can be written $2\log|a(0)|$, (4.6) becomes

$$J(\mathbf{a}) := \mathbb{J}_p(a^*a) = \mathbf{a}^*\mathbf{P}\mathbf{a} - 2\log|\mathbf{a}^*g(0)|.$$

Setting the gradient of $J(\mathbf{a})$ equal to zero, we obtain $\mathbf{a} = \mathbf{P}^{-1}g(0)/|a(0)|$ and hence $a(z) = g(z)^*\mathbf{P}^{-1}g(0)/|a(0)|$. Then $|a(0)|^2 = g(0)^*\mathbf{P}^{-1}g(0)$, and therefore the optimal $\mathbf{a}$ becomes

$$a(z) = \frac{g(z)^*\mathbf{P}^{-1}g(0)}{\sqrt{g(0)^*\mathbf{P}^{-1}g(0)}}. \tag{5.8}$$

Moreover, in view of Theorems 4.1 and 4.2,

$$\hat{\Phi}(t) = \frac{1}{Q(t)} = \frac{1}{|a(e^{it})|^2},$$

and therefore (5.7) follows from (5.8). □

In the trigonometric moment problem, modulo normalization,

$$\varphi_n(z) := g(z)^*\mathbf{P}^{-1}g(0)$$

reduces to the Szegö polynomial orthogonal on the unit circle of degree n (cf [10]).

6. Amplifications and conclusions

In this paper we showed that the moment problem for rational positive measures is solvable for all strictly positve sequences, provide Hypotheses 2.3 and 3.5 hold for $\mathfrak{P}$. In the language of functions and spaces, we showed that the moment map $\mathfrak{M}$ defined by (2.3) restricts to a surjection

$$\mathfrak{M}|_{\mathcal{R}_+} : \mathcal{R}_+ \to \mathring{\mathfrak{C}}_+ \tag{6.1}$$

by proving that the restriction

$$\mathfrak{M}|_{\mathcal{P}_+} : \mathcal{P}_+ \to \mathring{\mathfrak{C}}_+ \tag{6.2}$$

is surjective. Indeed, using the strict convexity of the dual functional, we were able to conclude in Corollary 4.3 that (6.2) is a bijection.

In this section we briefly discuss these maps in more detail. Following Hadamard, the problem of solving, for $c \in \mathring{\mathfrak{C}}_+$, the equations

$$\mathfrak{M}|_{\mathcal{P}_+}(d\mu(q)) = c, \text{ for } q \in \mathring{\mathfrak{P}}_+, \tag{6.3}$$

is *well-posed* provided a solution q exists, is unique and varies continuously with c (in some reasonable topology). As elements of open convex subsets of Euclidean space, the choice of topology is clear. Existence and uniqueness is the essence of Corollary 4.3. Moreover, our proof of Proposition 3.11 reposed on the observation that $\mathrm{Jac}(\mathfrak{M}|_{\mathcal{P}_+})_q$ is nonsingular at each $q \in \mathring{\mathfrak{P}}_+$ so that, by the Inverse Function Theorem, $\mathfrak{M}|_{\mathcal{P}_+}$ is a smooth bijection with a smooth inverse. Since $\mathfrak{M}|_{\mathcal{P}_+}^{-1}$ is differentiable, it is continuous, so that q is a continuous function of c and this restricted moment problem is well-posed.

Our second amplification concerns the map (6.1). Here, $\mathfrak{M}|_{\mathcal{R}_+}$ is not injective and one would instead like a continuous or smooth parameterization of the solutions, for $c \in \mathring{\mathfrak{C}}_+$, to the equations

$$\mathfrak{M}|_{\mathcal{R}_+}(d\mu) = c, \text{ for } d\mu \in \mathcal{R}_+. \tag{6.4}$$

As before, one can compute the Jacobian $\text{Jac}(\mathfrak{M}|_{\mathcal{R}_+})_{d\mu}$ and show [9] that

$$\text{rank Jac}(\mathfrak{M}|_{\mathcal{R}_+})_{d\mu} = 2n - r + 2,$$

for all $d\mu \in \mathcal{R}_+$. In fact, in [9] we prove that the solution space $\mathfrak{M}|_{\mathcal{R}_+}^{-1}(c)$ is a smooth manifold, smoothly parameterized by $p \in \mathring{\mathfrak{P}}_+$ as described in Theorem 4.1.

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