Homework Problems
Lectures on Moments and their Applications to Systems, Signals and Control Due Thursday in class.

1. a. Prove the Abel-Jacobi-Liouville Formula

$$\det \mathrm{e}^{A} = \mathrm{e}^{\mathrm{tr}(A)}$$

using a general position argument.

(Hint: View the two sides of the identity as two continuous functions, $F_1, F_2$ on the vector space $\mathcal{M}_n(\mathbb{C})$ of complex $n$ by $n$ matrices. Prove $F_1(D) = F_2(D)$ for any diagonal matrix $D \in \mathcal{M}_n(\mathbb{C})$ and complete the proof by using this to prove $F_1(M) = F_2(M)$ for a dense subset of matrices.)

b. Assume $A \in \mathcal{M}_n(\mathbb{R})$. Show that $\mathrm{tr}(A) = 0$ if, and only if, the flow of $\dot{x} = Ax$ preserves volume.

c. Show that a necessary condition for stability of the origin for $\dot{x} = Ax$ is that the flow decreases volume.

2. Let $\mathcal{S}_n$ denote the open set of degree $n$ real polynomials

$$a(z) = a_0 + a_1 z + \cdots + a_n z^n$$

with $n$ roots in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. A tangent vector to $\mathcal{S}_n$ at $a$ can be thought of as a real polynomial $b$ of degree $d \leq n$ in terms of the small perturbation vector $a + \epsilon b$ based at $A$. If $d(z)$ is a pseudo-polynomial of degree $n$ that is positve on the unit circle then it admits a spectral factorization there of the form

$$d(z) = a(z)a(1/z)$$

where $a \in \mathcal{S}_n$. Let's view spectral factorization as a function.

$$T(a) = a(z)a(1/z)$$

2a. Show that the directional derivative of $T$ in the direction $b$ at $a \in \mathcal{S}_n$ is given by

$$a(z)b(1/z) + a(1/z)b(z)$$

2b. Show that the Jacobian matirx of $T$ is invertible at every $a \in \mathcal{S}_n$.

3. Consider the Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0$$

where $(A, B, C)$ is a minimal triple. As you know, there always exists a unique positive definite solution $P_*(A, B, C)$. A tangent vector to $P_*$ in the space of positive definite matrices can be thought of as a symmetric matrix $Q$ leading to the small perturbation vector $P_* + \epsilon Q$ based at $P_*$.

3a. Compute the derivative of the function $F(A, B, C, P) = A^T P + PA - PBB^T P + C^T C$ at the point $(A, B, C, P_*)$ in the direction $Q$ and set it equal to 0. What kind of famous equation is this?

3b. Show that this directional derivative is 0 if and only if $Q = 0$

3c. Apply the Implicit Function Theorem to the equation $F(A, B, C, P) = 0$ at the point $(A, B, C, P_*)$ to show that $P_*(A, B, C)$ is an an analytic function of $(A, B, C)$.