Washington University in St. Louis

SCHOOL OF ENGINEERING & APPLIED SCIENCE

Lectures on Moment Problems in Signals, Systems and Control

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1. The power moment problem
2. Interpolation problems
3. Deterministic and Stochastic Partial Realizations
4. The generalized moment problem of Krein et al.
5. Markov's moment problem and time optimal control
6. The (generalized) moment problem for rational measures
7. A Dirichlet Principle for the moment problem with rational measures
8. The Covariance Extension Problem
9. Nevanlinna-Pick Interpolation for Rational Functions
10. Hadamard's Inverse Function Theorem

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- **Central Limit Theorem - Chebychev 1887**

  If X_i are identically distributed random variables with $E(X_i) = 0, \ \ \sigma^2_{X_i} = 1$, then

  $Y_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}} \to_D Y$ **where Y is $N(0,1)$**

Fourier methods in probability: the characteristic function

$$\phi_X(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{i\xi x} p(x) dx = \hat{p}(\xi)$$

$$1. \ \hat{p} = \hat{q} \iff p(x)dx = q(x)dx$$

$$2. \ X_n \to_D X \iff \phi_{X_n}(\xi) \to \phi_X(\xi)$$

Expanding the characteristic function in a Taylor series

$$\phi_X(\xi) = \hat{p}(\xi) = E(e^{i\xi X}) = \phi_X(0) + \phi'_X(0)\xi + \frac{\phi''_X(0)}{2!}\xi^2 + \ldots$$

$$\phi_X^{(k)}(0) = i^k E(X^k)$$

where $E(X^k) = \int_{-\infty}^{\infty} x^k p(x) dx$ **is the k-th moment of** X

Proof of CLT. 1. $\phi_{X_1}(\xi) = 1 - \xi^2/2 + o(\xi^2)$

2. $\phi_{Y_n}(\xi) = \phi_{X_1}(\xi/\sqrt{n}) \cdots \phi_{X_n}(\xi/\sqrt{n})$

3. $\log(\phi_{Y_n}(\xi)) = \sum_{i-1}^{i=n}(-\xi^2/2n + o(\xi^2/n))$
$\to -\xi^2/2 = log(\phi_Y(\xi))$

THE POWER MOMENTS OF A LINEAR SYSTEM

Similarly, if $H(s) = C(sI - A)^{-1}B + D$ is the transfer function of a linear systems (A, B, C, D), then the moments of H may be defined as

$$\eta_k = (-1)^k \frac{d^k H}{ds^k}(0)$$

If $\sigma(A) \subset \mathbb{C}^-$,

$$\eta_k = (-1)^k \frac{d^k H}{ds^k}(0) = \int_0^\infty t^k h(t) dt \text{ for } k > 0$$

where $h(t) = C\mathrm{e}^{At}B$, If $k = 0$, $\eta_0 = D$, the DC gain.

The Final Value Theorem implies that for any other stable linear system whose transfer function $K(s)$ satisfies

$$\frac{d^k K}{ds^k}(0) = (-1)^k \eta_k, \;\; 0 \le k \le d$$

the difference between the responses to a fixed polynomial input $u(t) = a_0 + \cdots + a_d t^d$ will decay as $t \to \infty$

If $\eta = (\eta_0, \ldots, \eta_d)$ is power moment sequence we say that η has length $\lambda(\eta) = d + 1$.

Theorem If η is a power moment sequence of length $d+1$, there for each Hurwitz polynomial $a(s) = a_0 + \cdots + a_{d-1}s^{d-1} + s^d$ there is a stable rational function $K(s) = b(s)/a(s)$ of degree $l \leq d$ such that

$$\frac{d^k K}{ds^k}(0) = (-1)^k \eta_k, \quad 0 \leq k \leq d.$$

In particular, if $s(\eta)$ denotes the minimal degree of a stable $K(s)$ which matches η, then $s(\eta) \leq \lambda(\eta) - 1$.

Proof. If $a(s)$ is any Hurwitz polynomial and $b(s) = b_0 + \cdots + b_{d-1}s^{d-1} + b_d s^d$ is defined by

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_d \end{pmatrix} = \begin{pmatrix} a_0 & 0 & \ldots & 0 \\ a_1 & a_0 & \ldots & 0 \\ \vdots & & & \\ 1 & a_{d-1} & \ldots & a_0 \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_d \end{pmatrix}$$

then $K(s) = \dfrac{b(s)}{a(s)}$ has the moment sequence $\eta_0, \ldots, \eta_d$

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In particular, if $s(\eta)$ denotes the minimal degree of a stable $K(s)$ which matches η, then $s(\eta) \leq \lambda(\eta) - 1$.

Remark. For $d = 2$, there are open sets $U_i = \{(\eta_0, \eta_1, \eta_2)\} \subset \mathbb{R}^3$, for $1 = 1, 2$ such that $s(\eta) = i$ for $\eta \in U_i$. In fact, $(1, -1, 1) \in U_1$ and $(1, -1, -3) \in U_2$

Remark. For $d = 2r$ or $d = 2r + 1$, there are open sets $U_i \subset \mathbb{R}^{d+1}$, for $1 = 1, 2$ such that $s(\eta) = r$ for $\eta \in U_1$ and $s(\eta) > r$ for $\eta \in U_2$

Model Reduction for Linear Systems (c. 1980: Fortman and Hitz, Kailath, Mahmoud and Singh; Antoulas (c. 2005), . . .)

$$\eta_k = \int_0^\infty t^k h(t)dt = (-1)^k \frac{d^k H}{ds^k}(0)$$

One approach to model reduction for $H(s)$ might be to find $K(s)$, with $\deg(K) << \deg(H)$, matching he first $d+1$ power moments of $H(s)$.

Padé approximation is a reduced order model obtained by matching η_k, for $k = 0, \ldots, \tilde{n} < n$.

A Padé approximant will reproduce the system response to polynomials, *provided the Padé approximant is stable.*

More generally, consider matching the "generalized" moments:

$$\eta_k(s_0) = \int_0^\infty t^k h(t)\mathrm{e}^{-s_0 t}dt = (-1)^k \frac{d^k H}{ds^k}(s_0)$$

For $s_0 = j\omega_0$, $\eta_0(s_0)$ is the frequency response.

In general, η_k, for $k = 0, \ldots, N$ determines the response to $p(t)\mathrm{e}^{js_0}$ where $\deg(p) \leq N$.

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- General Interpolation Problem: Given $(n+1)$ points $\{z_k\}_{k=0}^{n} \subset \mathcal{R}$, a region of $\mathbb{C}$, and a sequence $\{w_{k_i}\}_{k=0,i=0}^{m,j_k}$ of desired values, find all meromorphic functions f in a given class $\mathcal{C}$ which satisfy

$$f^{(i)}(z_k) = w_{k_i}, \quad k = 0, 1, \cdots, m; \quad i = 0, \cdots, j_k;$$

$$\sum_{k=0}^{m} j_k = (n+1)$$

- If $\mathcal{R} = \mathbb{C}$ and $\mathcal{C}$ is the class of polynomials, this is Lagrange interpolation.

- **If $\mathcal{R} = \mathbb{C}$ and $\mathcal{C}$ is the class of rational functions, this is the rational interpolation problem.**

- **If $\mathcal{R} = \overline{\mathbb{C}^+}$ and $\mathcal{C}$ is the class of rational functions, this is the stable rational interpolation problem.**

- (Carathéodory, Nevanlinna, Pick, Toeplitz, Schur) Find all positive real interpolants. Here $\mathcal{R} = \{z : |z| \geq 1\}$ and $\mathcal{C}$ is the class of all functions satisfying $\mathrm{Re}\, f(z) \geq 0$ for $|z| \geq 1$.

- If $\mathcal{R} = \overline{\mathbb{C}^+}$ and $\mathcal{C}$ is the class of rational functions, this is the stable rational interpolation problem.

We now consider interpolation data $\eta = \{(z_i, w_i)\}_{i=1}^{i=n+1}$ of length $\lambda(\eta) = n+1$, with $w_i \neq w_j$ for $i \neq j$

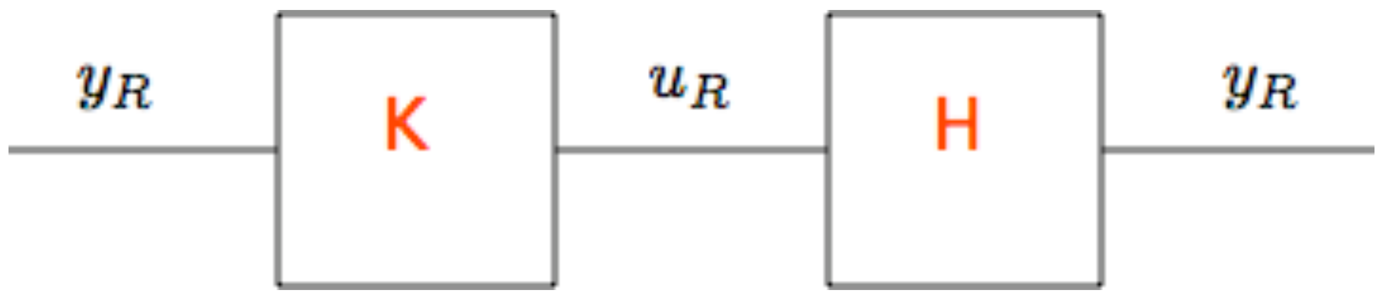
Theorem If η is a self-conjugate interpolation string of length $n+1$, then for each real Hurwitz polynomial $a(s) = a_0 + \cdots + a_{n-1}s^{n-1} + s^n$ there is a stable, real rational function $K(s) = b(s)/a(s)$ of degree $l \leq n$ such that

$$\frac{K}{ds}(z_k) = w_k, \quad 0 \leq k \leq n+1$$

In particular, if $s(\eta)$ denotes the minimal degree of a stable $K(s)$ which interpolates η, then $s(\eta) \leq n$.

Proof. If $a(s)$ is any Hurwitz polynomial and $b(s) = b_0 + \cdots + b_{d-1}s^{d-1} + b_d s^d$ is the Lagrange interpolation polynomial solving $b(z_j) = w_j/a(z_j)$, then $K(s) = \dfrac{b(s)}{a(s)}$ interpolates the data string η

Reduced order controllers for causal or noncausal compensators



Example. Consider the system (Curtain and Zwart)

$$z_t(x,t) = z_{xx}(x,t) \tag{1}$$
$$z(0,t) = 0, \tag{2}$$
$$z_x(1,t) = u \tag{3}$$
$$z(x,0) = \varphi(x). \tag{4}$$
$$y(t) = z(1,t), \tag{5}$$

$$H(s) = \frac{\sinh(\sqrt{s})}{\sqrt{s}\cosh(\sqrt{s})}$$

For $y_R(t) = A\sin(\omega t + \phi)$, why not solve the rational interpolation problem $K(j\omega) = H^{-1}(j\omega)$?

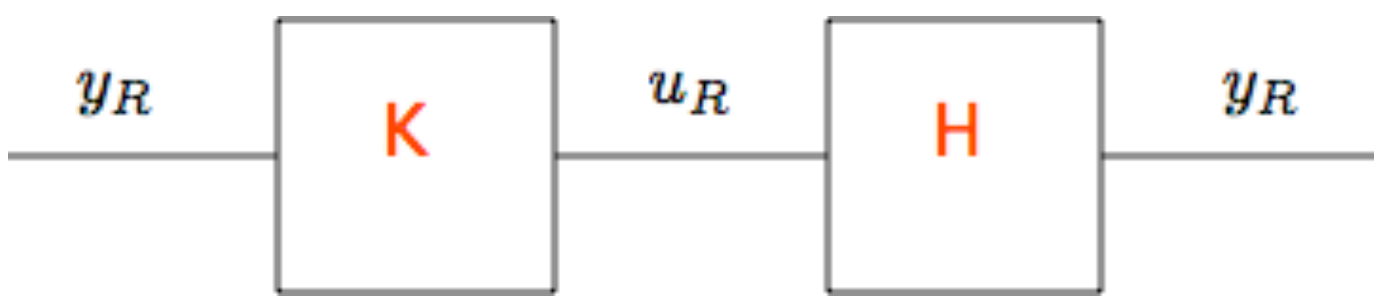
$$H(s) = \frac{\sinh(\sqrt{s})}{\sqrt{s}\cosh(\sqrt{s})}$$

$H^{-1}(s)$ is the transfer function of

$$\begin{aligned} &z_t(x,t) = z_{xx}(x,t) \\ &z(1,t) = w(t) \\ &z(1,t) = 0 \\ &z(x,0) = 0. \end{aligned}$$

with input $w(t)$ and output $v(t) = z_x(1,t)$.

This is the zero dynamics of the control system.



Consider $y_R(t) = \sin(2t)$.

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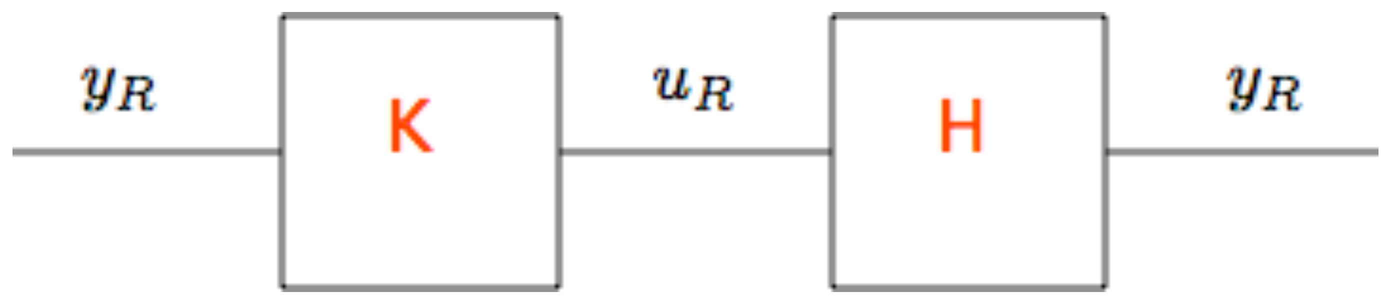


For the cascade compensator we can choose either H^{-1}:

$z_t(x,t) = z_{xx}(x,t)$
$z(1,t) = w(t)$
$z(1,t) = 0$
$z(x,0) = 0$. with input $w(t) = \sin(2t)$ and output $v(t) = z_x(1,t)$.

or

$K(s) = K\frac{(s+\beta)}{(s+1)}$, where β and K are chosen to tune the phase and magnitude of $K(2j)$.

Since the reference trajectory is $y_R(t) = A\sin(2t)$, we have interpolation data of length 2

$$K(2i) = H^{-1}(2i) = 1.0856 + 0.6504i,$$

and

$$K(-2i) = H^{-1}(-2i) = 1.0856 - 0.6504i,$$

rounding to four decimals.

There exists a stable degree 1 interpolating compensator, for example

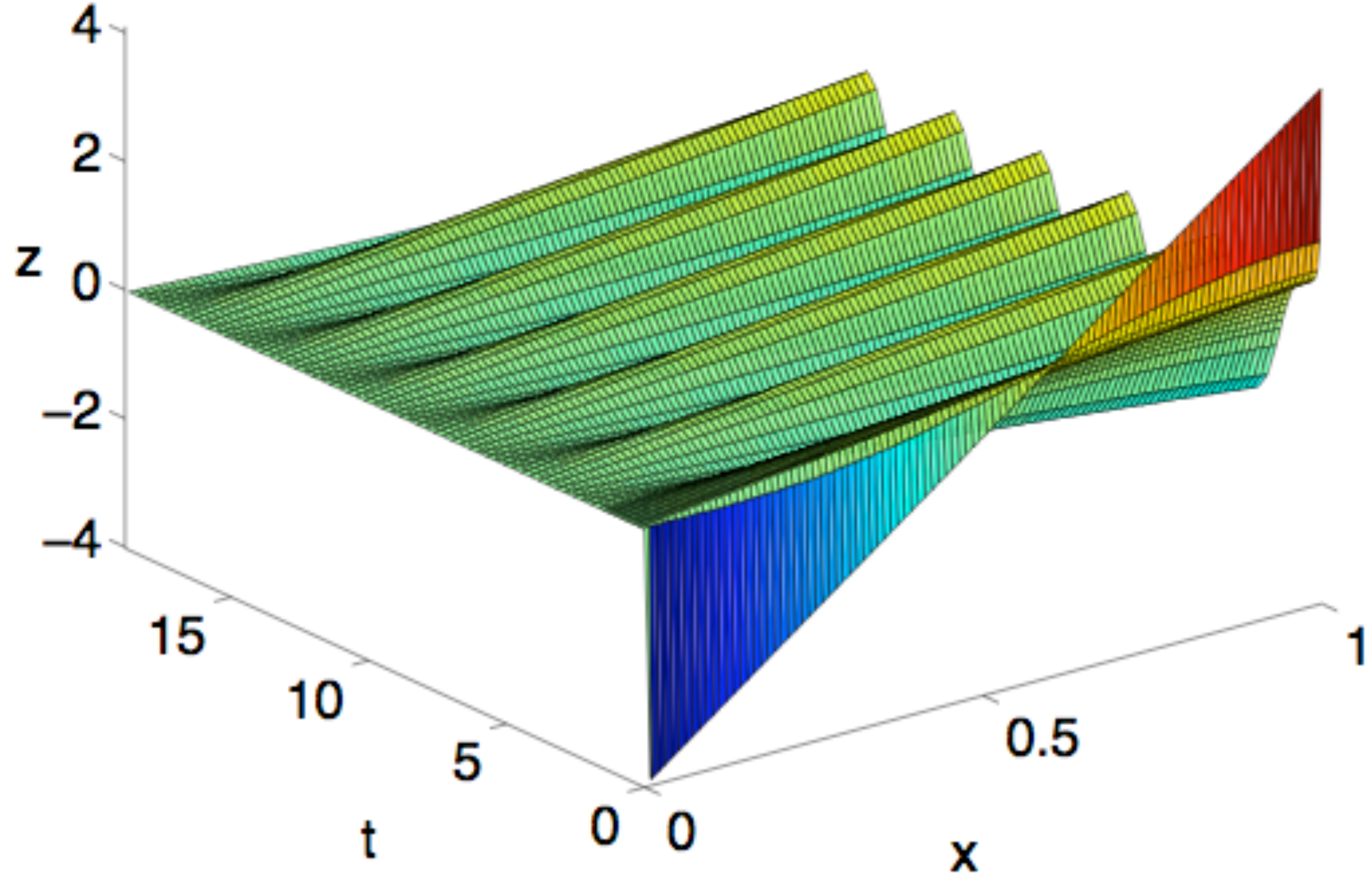
$$K(s) = 1.4108\,\frac{s - .1525}{s + 1}$$

Indeed, driving K with $y_r(t)$ produces the steady-state control law

$$u_R(t) = 1.2655\sin(2t + 0.5397)$$

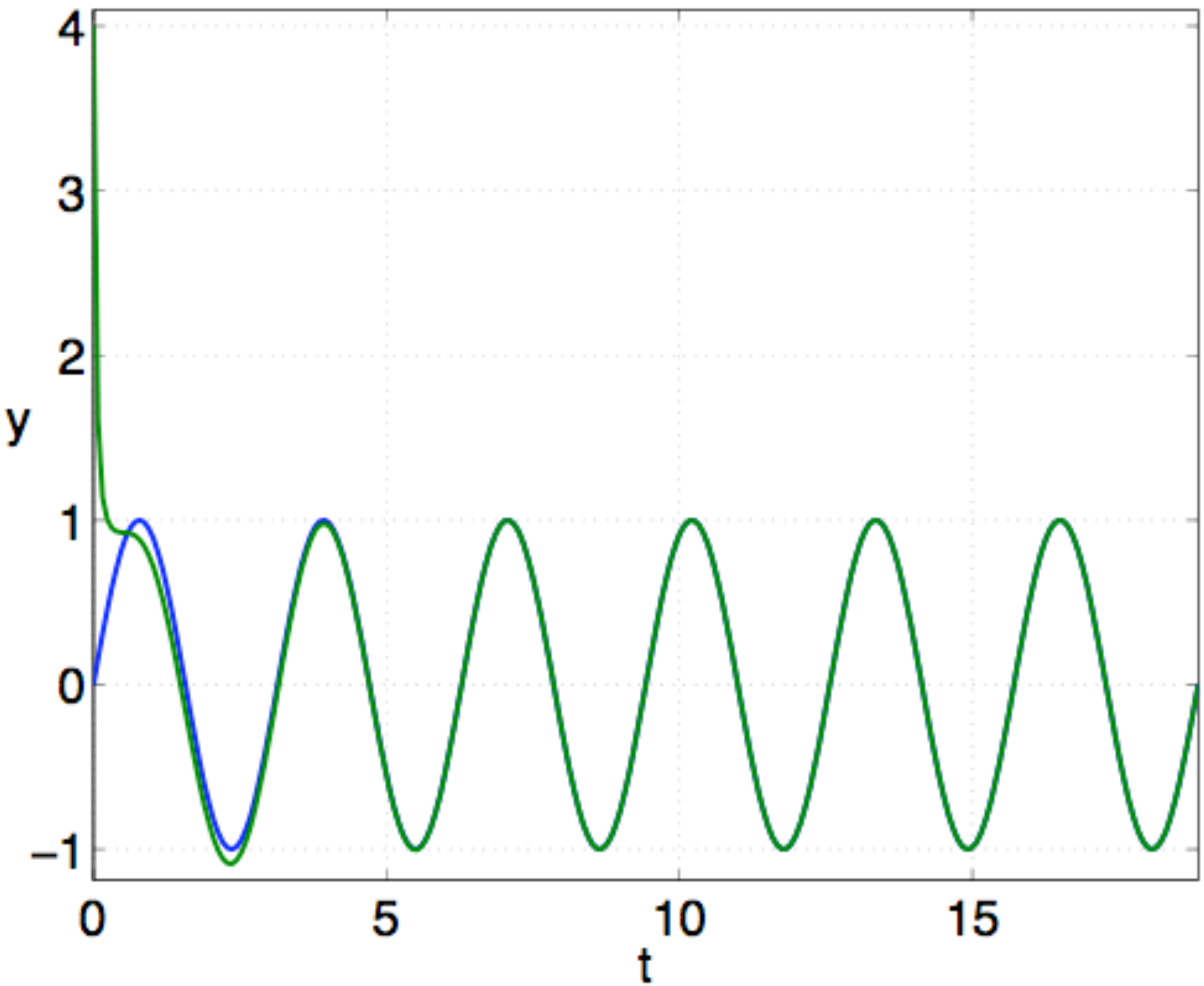
In the following simulations, we have taken initial condition $\varphi(x) = -4(1 - 2x)$.

The steady state behavior of the state trajectory is illustrated in the next figure.



$$\varphi(x) = -4(1-2x)$$

The next figure depicts the controlled output trajectory and the trajectory to be tracked.



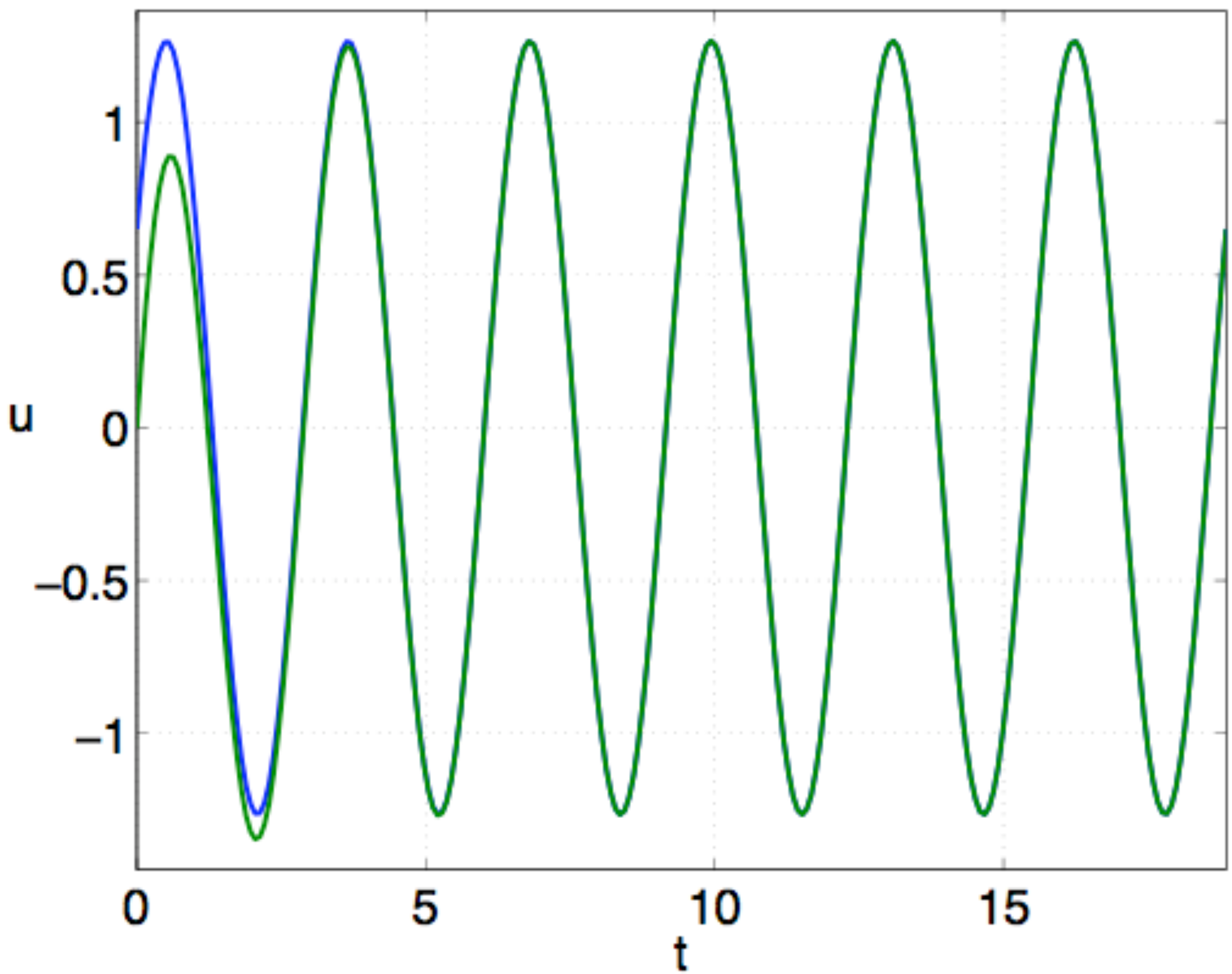
Finally, we know that the desired control can also be obtained by solving the regulator equations of output regulation theory for DPS.

In contrast to our classical design methods, the output regulator equations "boil down" to a linear parabolic PDE in two spatial dimensions.

In this case we know that the control can be written as

$$u_{\mathsf{RE}}(t) = \textbf{Re}\ \big(g(i\omega)^{-1}\big) M \sin(\omega t) + \textbf{Im}\ \big(g(i\omega)^{-1}\big) M \cos(\omega t)$$

In the next figure, we compare $u_R(t)$ with the control $u_{RE}(t)$ obtained from solving the regulator equations.



1
0.5
u
0
−0.5
−1
0
5
10
15
t

Theorem If η is a self-conjugate interpolation string of length $n+1$, then for each real Hurwitz polynomial $a(s) = a_0 + \cdots + a_{n-1}s^{n-1} + s^n$ there is a stable, real rational function $K(s) = b(s)/a(s)$ of degree $l \leq n$ such that

$$\frac{K}{ds}(z_k) = w_k, \quad 0 \leq k \leq n+1$$

In particular, if $s(\eta)$ denotes the minimal degree of a stable $K(s)$ which interpolates η, then $s(\eta) \leq n$.

Remark. For $n = 2$, there are open sets $U_i \subset \mathbb{R}^3$, for $1 = 1, 2$ of interpolation strings η such that $s(\eta) = i$ for $\eta \in U_i$.

Example 2. Consider the critically damped harmonic oscillator

$$H(s) = \frac{1}{s^2 + 2s + 1}$$

and the induced one-parameter family of interpolation problems

$$K_\epsilon(2i+\epsilon) = H(2i+\epsilon),\; K_\epsilon(-2i+\epsilon) = H(-2i+\epsilon) \;\; K_\epsilon(\infty) = H(\infty) = 0$$

for a first order, stable interpolant K_ϵ.

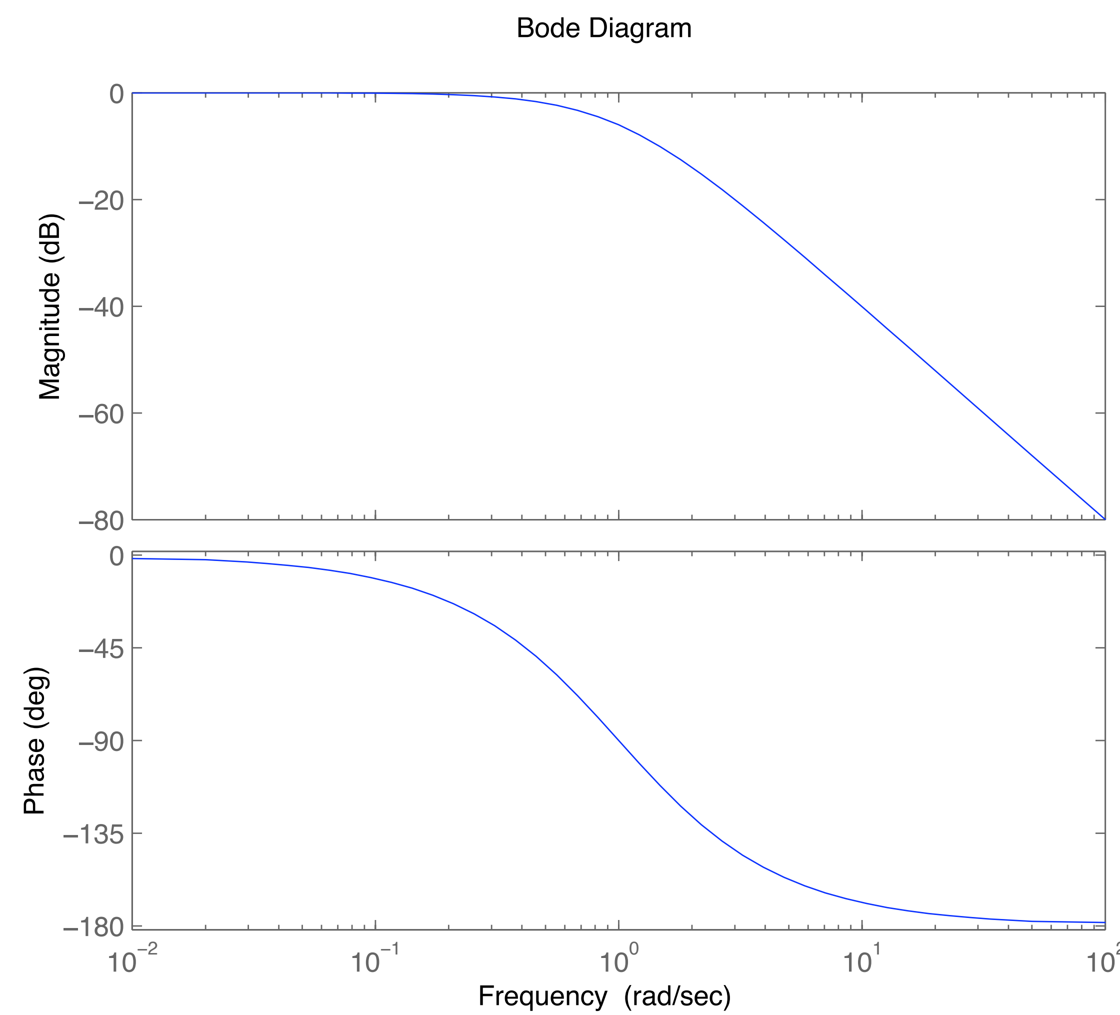
$$H(s) = \frac{1}{s^2 + 2s + 1}$$

$$K_\epsilon(2i+\epsilon) = H(2i+\epsilon),\ K_\epsilon(-2i+\epsilon) = H(-2i+\epsilon)\ \ K_\epsilon(\infty) = H(\infty) = 0.$$

Note Bene. 1. $-\pi/2 < \angle K(i\omega) < 0$ for any stable, strictly proper K with a positive high-frequency gain,

2. $\pi/2 < \angle K(i\omega) < \pi$ for any stable, strictly proper K with a negative high-frequency gain,

3. $-\pi < \angle H(i\omega) < -\pi/2$ for $\omega > 1$



Example 3. Consider the stable, minimum phase system

$$H_\epsilon(s) = \frac{s+1+\epsilon}{s^2+2s+1}$$

and the induced one-parameter family of interpolation problems

$$K_\epsilon(i) = H_\epsilon(i+\epsilon),\; K_\epsilon(-i) = H_\epsilon(-i+\epsilon),\; K_\epsilon(\infty) = H_\epsilon(\infty) = 0,$$

for a first order, stable interpolant K_ϵ.

Note Bene 1. $\epsilon = 0, K_0(s) = \dfrac{1}{s+1}$

2. If $-1 < \epsilon < 1$, then $-\pi/2 < \angle H_\epsilon(i\omega) < 0$

3. There exists a stable first order interpolant K_ϵ.

- General Interpolation Problem: Given $(n+1)$ points $\{z_k\}_{k=0}^{n} \subset \mathcal{R}$, a region of $\mathbb{C}$, and a sequence $\{w_{k_i}\}_{k=0,i=0}^{m,j_k}$ of desired values, find all meromorphic functions f in a given class $\mathcal{C}$ which satisfy

$$f^{(i)}(z_k) = w_{k_i}, \quad k = 0, 1, \cdots, m; \quad i = 0, \cdots, j_k;$$

$$\sum_{k=0}^{m} j_k = (n+1) \qquad .$$

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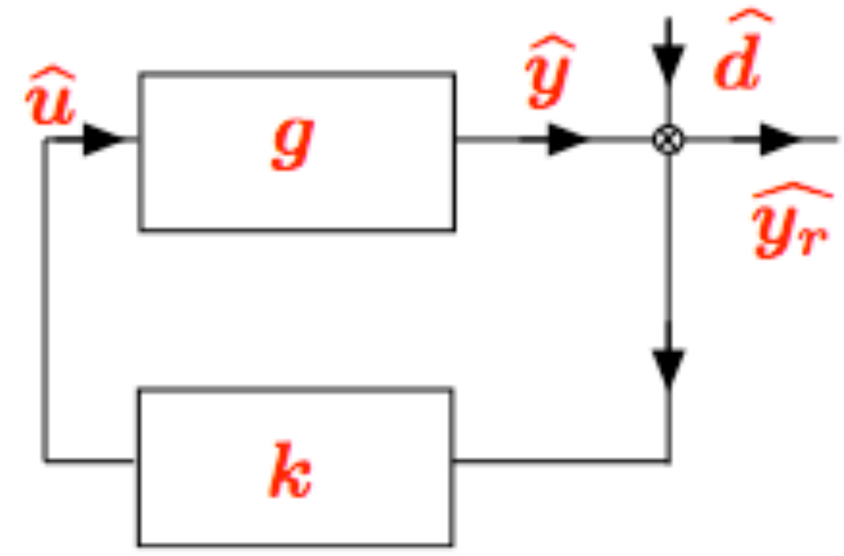
- **If $\mathcal{R} = \mathbb{C}$ and $\mathcal{C}$ is the class of rational functions, this is the rational interpolation problem.**

- **If $\mathcal{R} = \overline{\mathbb{C}^{+}}$ and $\mathcal{C}$ is the class of rational functions, this is the stable rational interpolation problem.**

- (Carathéodory, Nevanlinna, Pick, Toeplitz, Schur) Find all positive real interpolants. Here $\mathcal{R} = \{z : |z| \geq 1\}$ and $\mathcal{C}$ is the class of all functions satisfying $\mathrm{Re}\, f(z) \geq 0$ for $|z| \geq 1$.

Sensitivity to Disturbances

- Suppose $g(s)$ has right half-plane poles at $p_1, \cdots, p_r$.
- Right half-plane zeros at $z_1, \cdots, z_\ell$



$$\widehat{y}_r(s) = S(s)\widehat{d}(s),$$

where $S(s) = (1 - g(s)k(s))^{-1}$

$\hat{d}$ **a disturbance**

- Internal Stability of Feedback System $\Leftrightarrow$
  - (i) S analytic in $\mathrm{Re}(s) \geq 0$
  - (ii) $S(p_i) = 0$
  - (iii) $S(z_k) = 1$

Conformal Equivalents of the Function Class $\mathfrak{C}$ for Nevanlinna-Pick Interpolation

- Bounded Real (Discrete-time) $g(z)$ rational, $\deg\,(g) \leq n$, analytic in $|z| \geq 1$ and, for $\mathbb{D}^c = \{z\,:\,|z| > 1\}$

$$g\,:\,\mathbb{D}^c \to \mathbb{D}$$

- Bounded Real (Continuous-time) $g(s)$ rational, $\deg\,(g) \leq n$ analytic for $\mathrm{Re}(s) \geq 0$ and

$$g\,:\,\mathbb{C}^+ \to \mathbb{D}$$

- Bounded Real (Continuous-time) $g(s)$ rational, $\deg(g) \leq n$ analytic for $\mathrm{Re}(s) \geq 0$ and

$$g : \mathbb{C}^+ \to \mathbb{D}$$

- This is useful for robust control since for $g(s) = c(sI - A)^{-1}b$

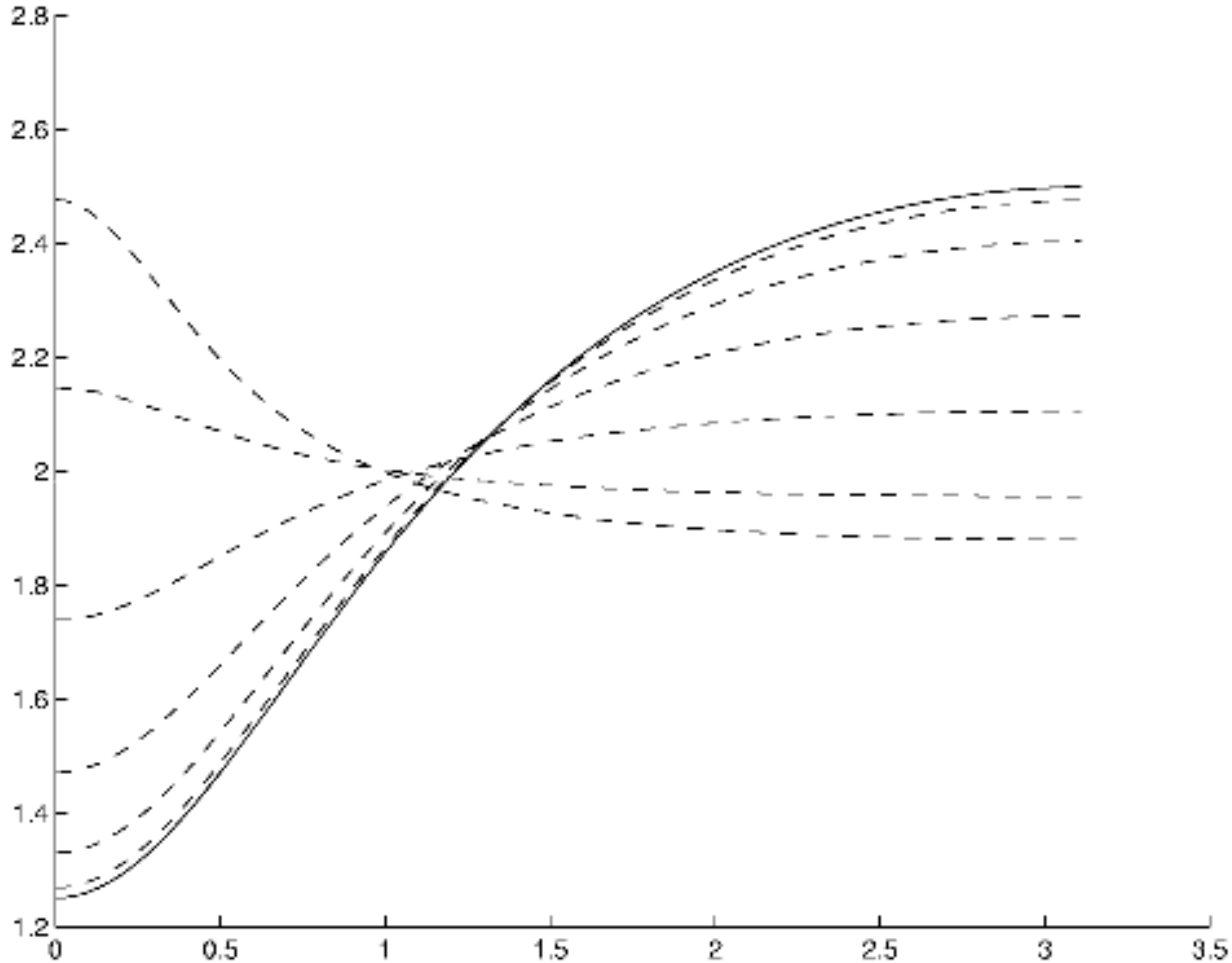
$$\dot{x} = Ax + bu$$

$$y = cx$$

$$\|y\|_2 \leq \|g\|_\infty \cdot \|u\|_2 \qquad \text{where } \sup_{jw} |g(jw)| = \|g\|_\infty \leq 1$$

- Internal Stability of Feedback System $\Leftrightarrow$
  - (i) S analytic in $\mathrm{Re}(s) \geq 0$
  - (ii) $S(p_i) = 0$
  - (iii) $S(z_k) = 1$
- Nevanlinna-Pick Interpolation for S bounded real, rational.

Degree 1 sensitivity functions for $G(z) = 1/(z-2)$



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