Washington University in St.Louis

SCHOOL OF ENGINEERING & APPLIED SCIENCE

Lectures on Moment Problems in Signals, Systems and Control

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Consider a subspace $\mathfrak{P}$ of the space $C(\mathcal{I})$ of complex-valued continuous functions on an interval $\mathcal{I} \subset \mathbb{R}$ and a choice of basis $(u_0, u_1, \cdots, u_n)$ of $\mathfrak{P}$

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Example 1. In a power moment problem,

$$u_k(t) = t^k, \quad k = 0, 1, \cdots, n,$$

and every $u \in \mathfrak{P}$ is a polynomial of degree $d \leq n$.

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Example 2. In the moment problem formulation of interpolation for distinct interpolation points $z_0, z_1, \ldots, z_n$ with $|z_k| < 1$, the basis functions are

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Indeed, if f is analytic inside $\mathbb{D}$ (and continuous on the boundary of $\mathbb{D}$), then

$$\int_{-\pi}^{\pi} f(t)u_k(t)dt = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{f(z)}{z-z_k}dz = f(z_k)$$

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Typical restrictions on the class of measures $d\mu$ include positivity, rationality, stability, minimum phase, and positive or bounded real and give rise to whole classes of constrained generalized moment problems

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Padé approximation is a reduced order model obtained by matching η_k, for $k = 0, \dots, \tilde{n} < n$.

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Denote by $d(c)$ **the smallest degree of an interpolating** f **and by** $s(c)$ **the smallest degree of a stable interpolating** f**.**

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Remark. For $d = 2r$ or $d = 2r + 1$, there are open sets $U_i \subset \mathbb{R}^{d+1}$, for $1 = 1, 2$ such that $s(\eta) = r$ for $\eta \in U_1$ and $s(\eta) > r$ for $\eta \in U_2$

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$$c_i = E(y_t y_{t+i}), \quad i = 0, \cdots, n, \cdots$$

find a filter (or dynamical I/O system) which "shapes" white noise into a process with the given (or desired) correlation coefficients.

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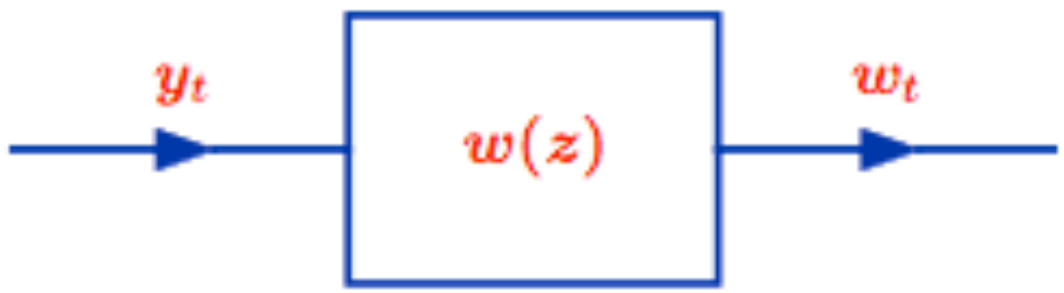

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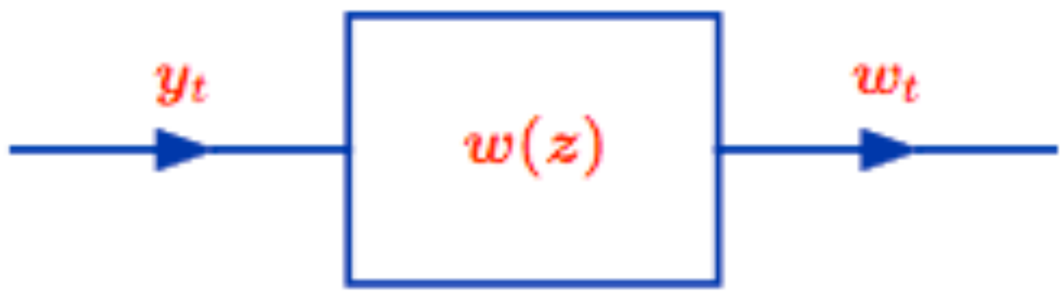

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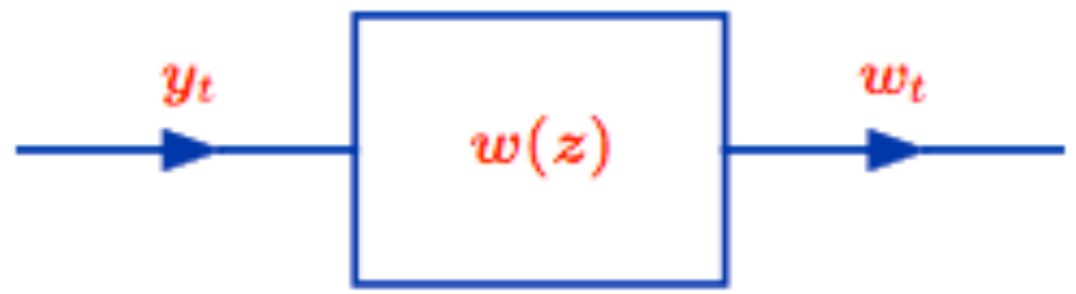

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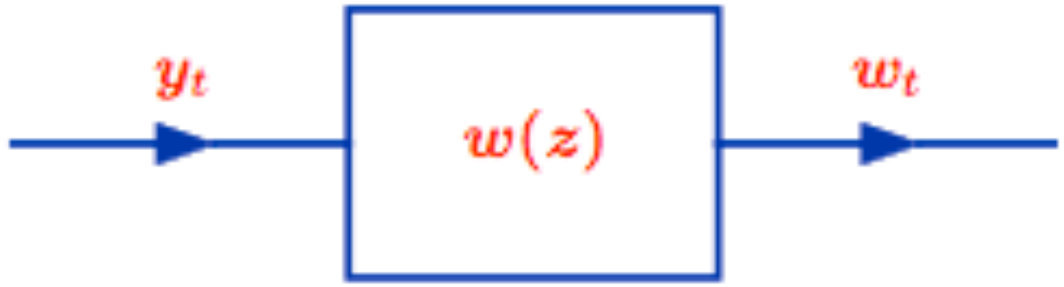

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then $\Phi(e^{i\theta})$ is positive if, and only if, $T_j > 0$, for all j

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Caratheodory: The trigonometric moment problem

Given real numbers $c_0, c_1, \cdots, c_n$ find all positive functions $\Phi(z)$ on the unit circle, harmonic in a neighborhood of the circle, with Fourier expansion

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MOMENT PROBLEM: Find Φ

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Modeling speech: The stochastic partial realization problem

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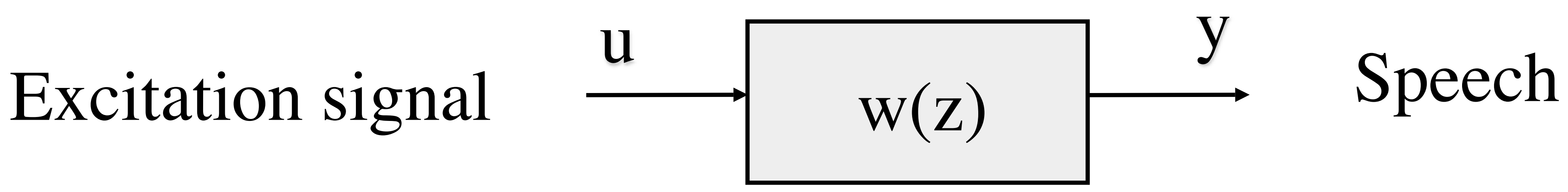

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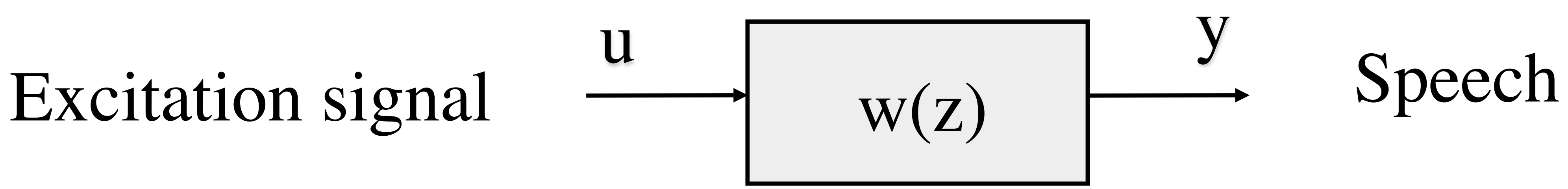
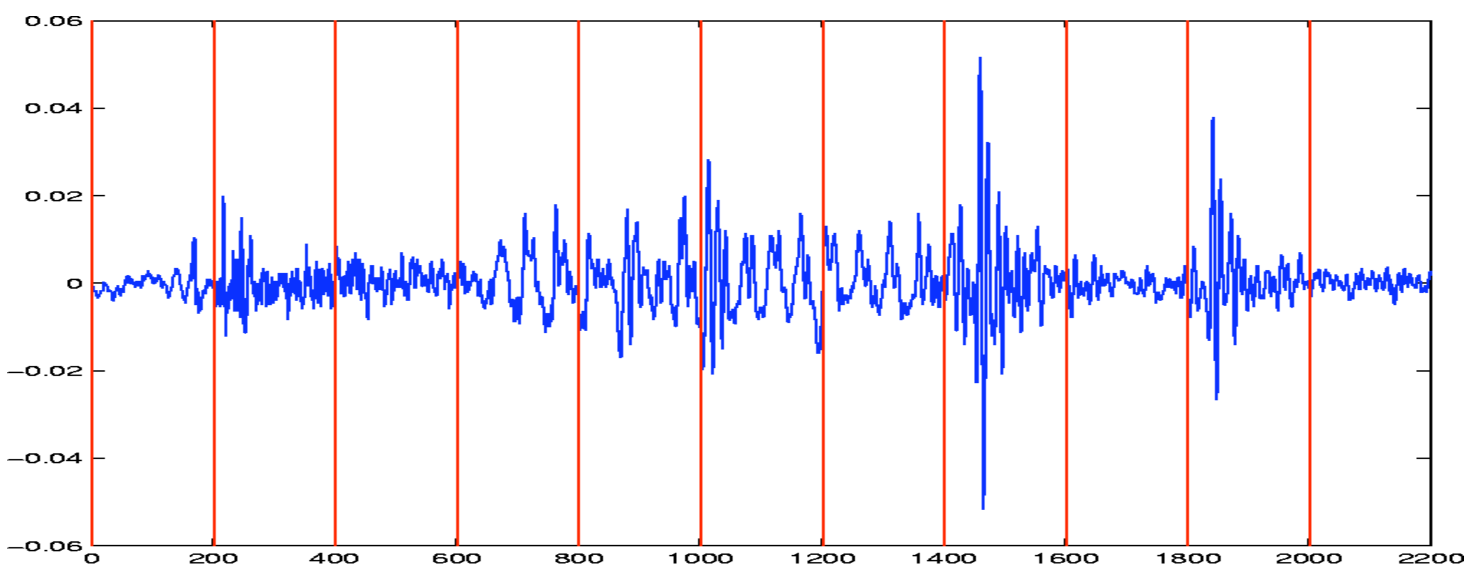
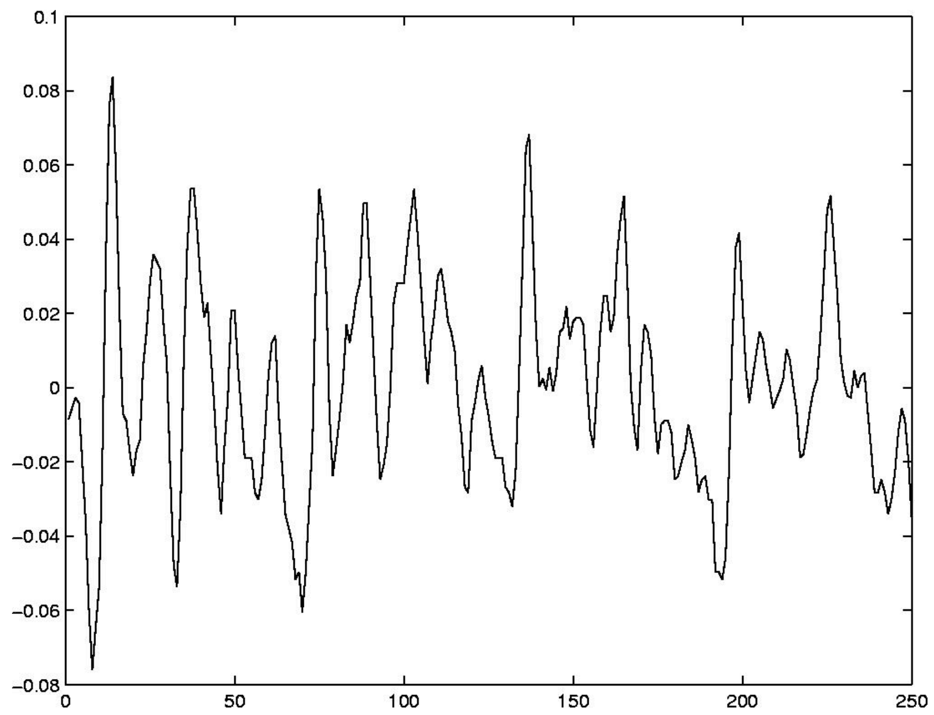




w(z) stationary on each (30 ms) subinterval

Spectral estimation from data



A 30 ms frame of speech for the voiced nasal phoneme [ng]

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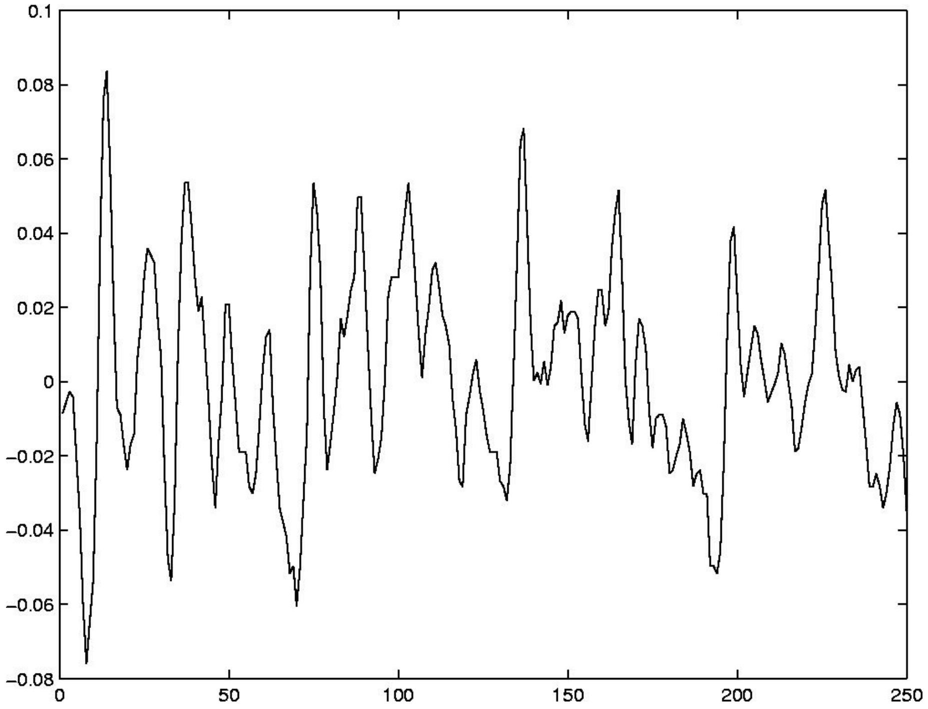

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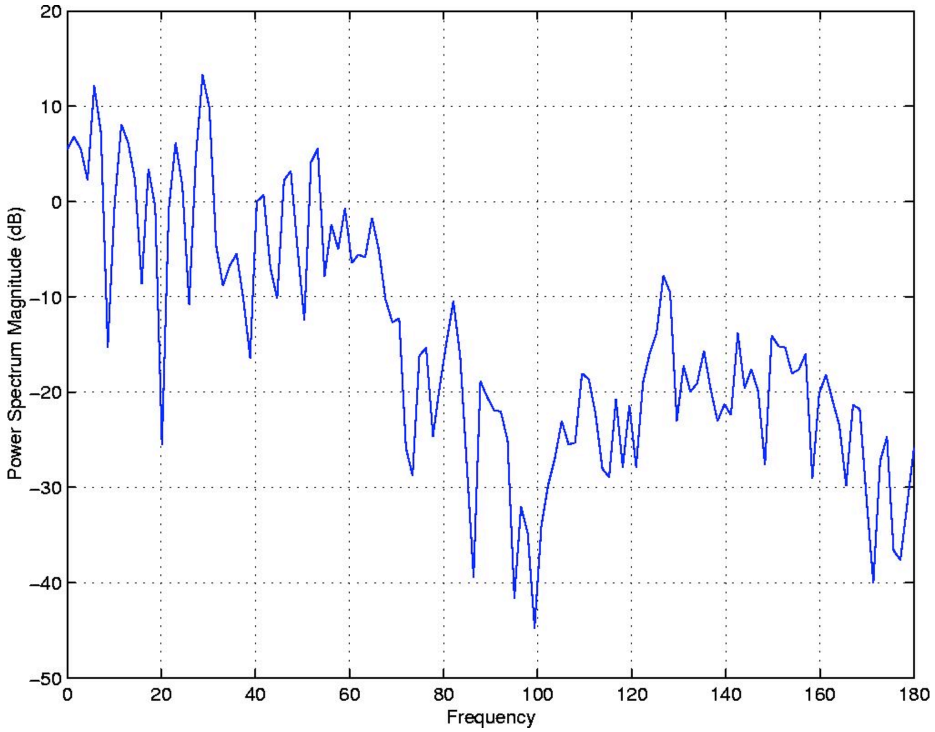
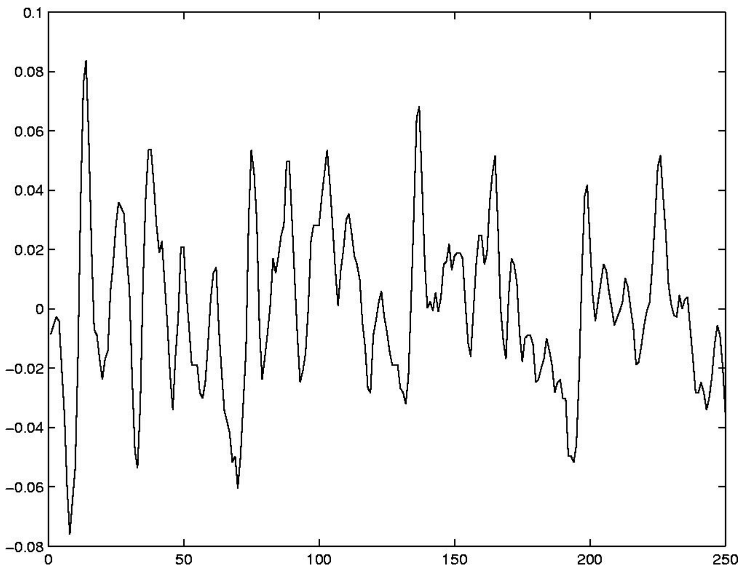
Estimate of spectral density:

Periodogram (FFT) of voiced nasal phoneme [ng]



Covariance estimates

Observed data: $y_0, y_1, \ldots, y_N$



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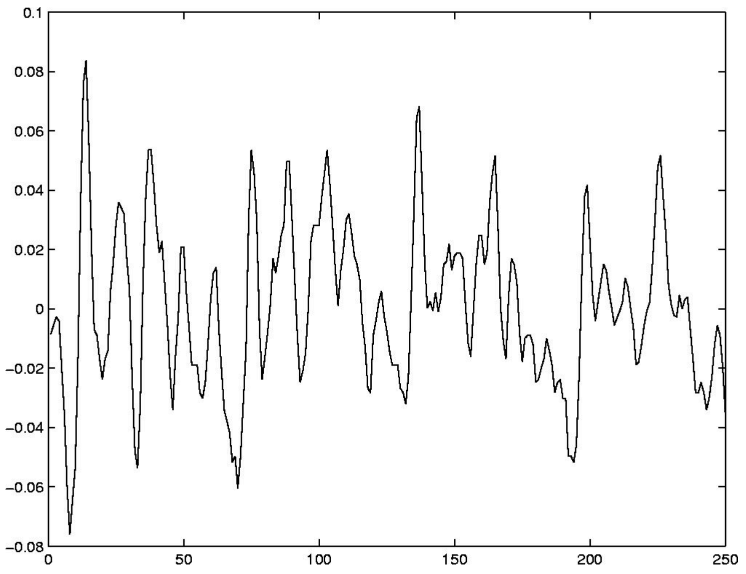

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We therefore estimate $c_0, c_1, \ldots, c_n$ where n << N

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yields Szegö polynomial

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and modeling filter

$$\boxed{w(z) = \frac{\sqrt{\rho_n} z^n}{\varphi_n(z)}} \qquad \text{where} \qquad \rho_n = \sum_{j=0}^{n} c_j \varphi_{nj}$$

LPC spectral envelope

Spectral envelope
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$$\Phi(e^{i\theta}) = \frac{\rho_n}{|\varphi_n(e^{i\theta})|^2}$$

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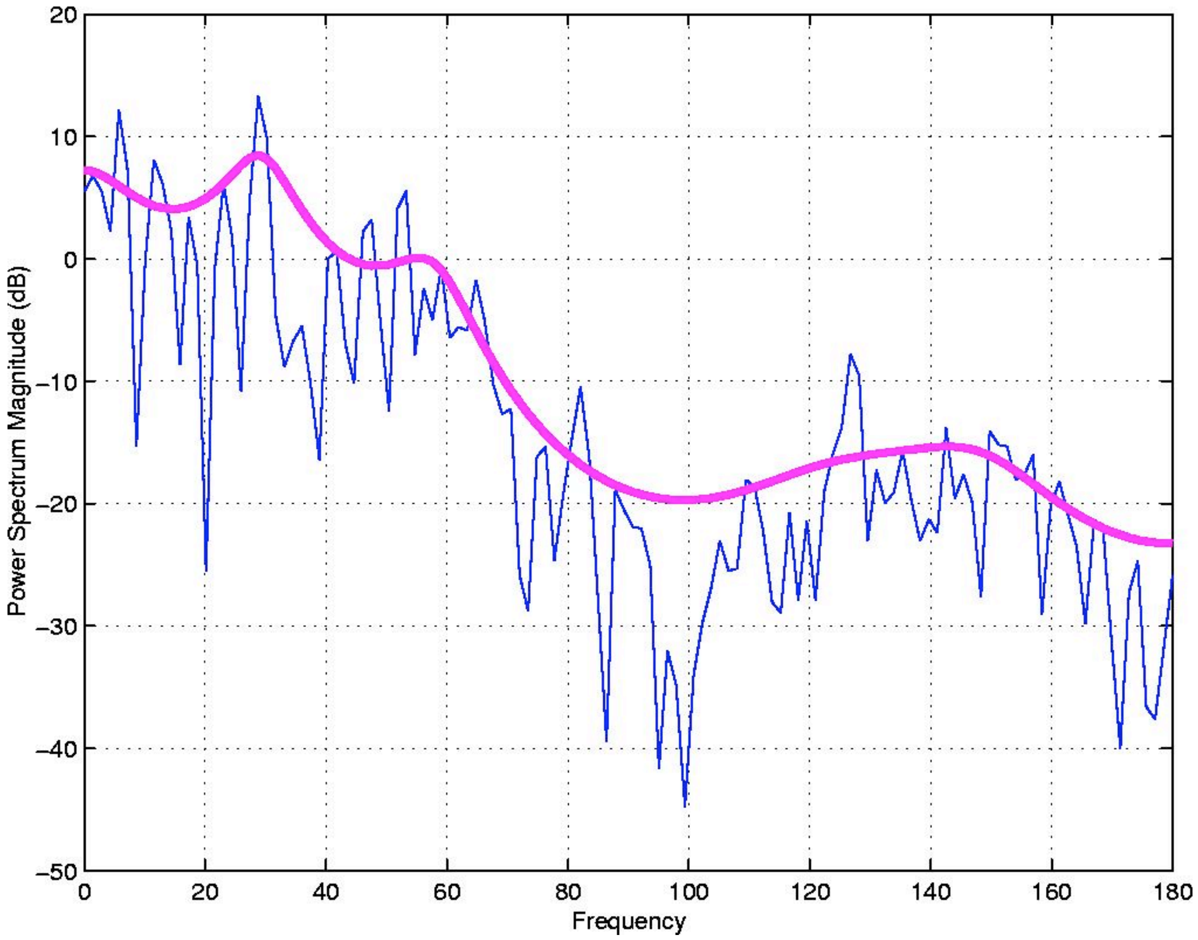

10th order LPC filter

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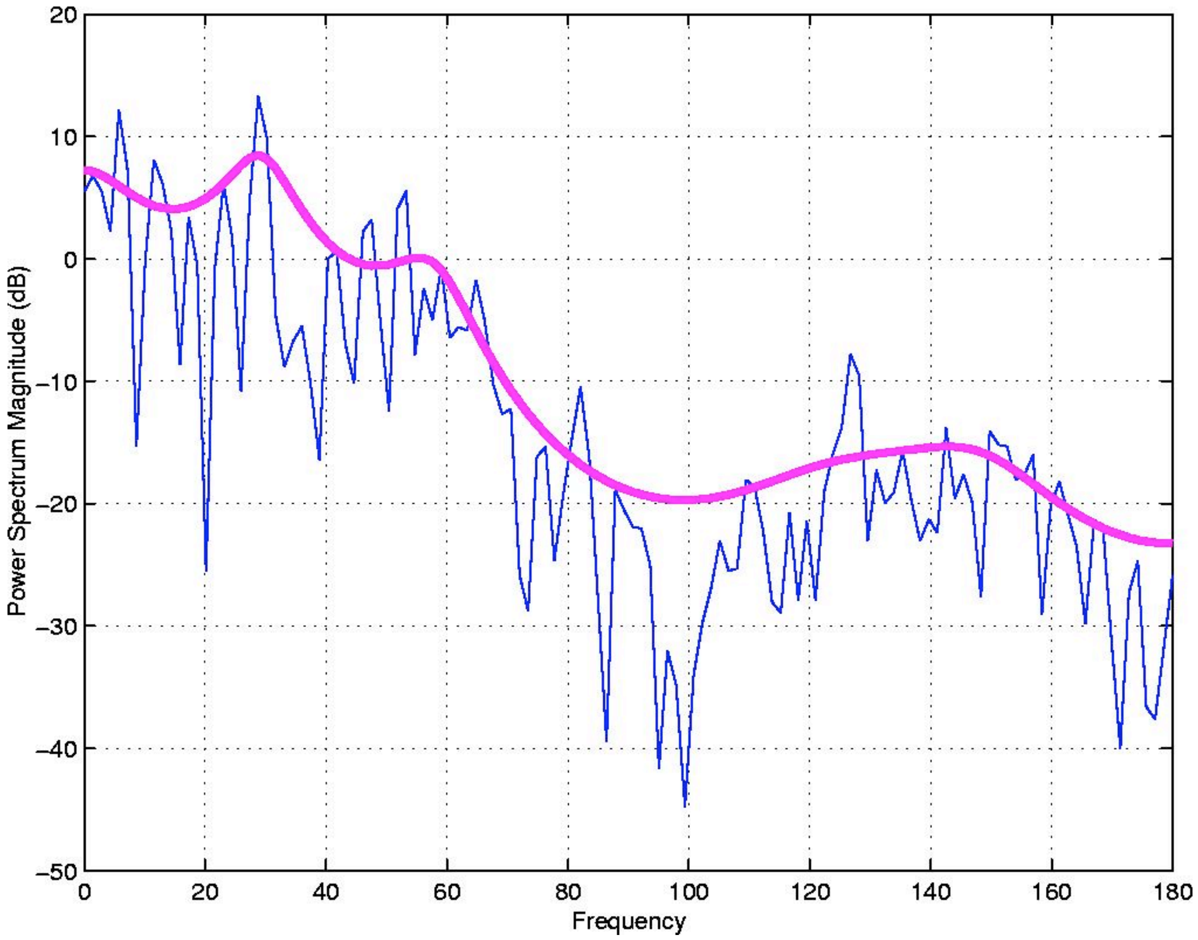

10th order LPC filter

- yields flat spectrum,
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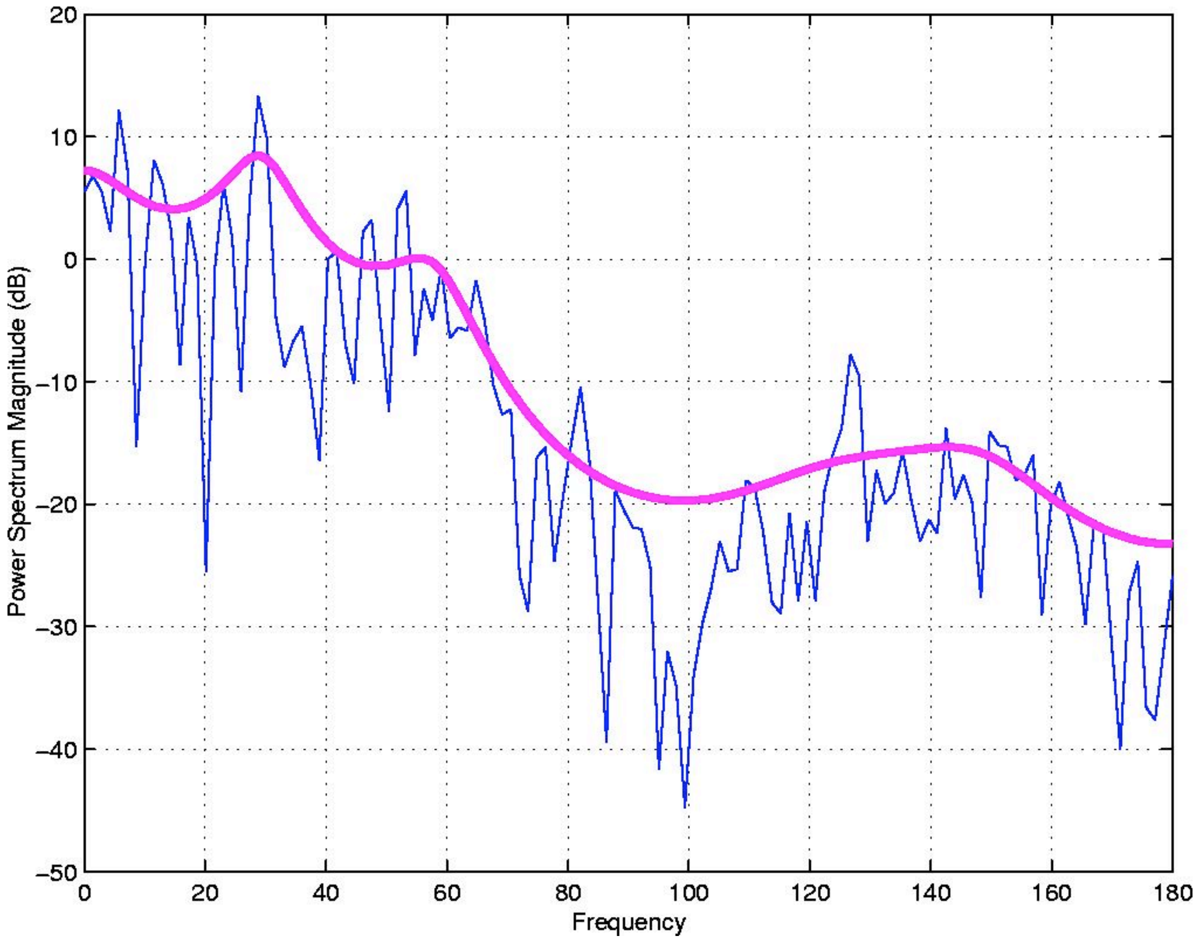

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Are there other, and better, solutions?

Caratheodory: The trigonometric moment problem

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MOMENT PROBLEM: Find Φ of (degree at most 2n) such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\theta = c_k, \quad k = 0, 1, \ldots, n$$

1. The power moment problem

2. Interpolation problems

3. Deterministic and Stochastic Partial Realizations

4. The generalized moment problem of Krein et al.

5. Markov’s moment problem and time optimal control

6. The (generalized) moment problem for rational measures

7. A Dirichlet principle for the moment problem with rational measures

8. The Covariance Extension Problem

3. Nevanlinna-Pick Interpolation for Rational Functions

• Hadamard’s Inverse Function Theorem

Consider a subspace $\mathfrak{P}$ of the space $C(\mathcal{I})$ of complex-valued continuous functions on an interval $\mathcal{I} \subset \mathbb{R}$ and a choice of basis $(u_0, u_1, \cdots, u_n)$ of $\mathfrak{P}$

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$\mathfrak{C}_+$ is a closed, convex cone, with $\mathfrak{C}_+^{\top} = \mathfrak{P}_+$.

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