



KTH Matematik

**Tentamen i 5B1872 Optimal Control**  
**Saturday March 9 2006 8.00–13.00**  
**Answers and solution sketches**

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1. (a)  $\lambda(T) = \begin{bmatrix} 1 \\ 1 \\ \text{free} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

(b) There is no feasible solution, hence  $= \infty$

(c)  $J = 1$

(d) The optimal control for the unconstrained problem,  $u(t) = -\frac{x(t)}{3-2t}$ , satisfies the constraint  $|u(t)| \leq 1$  and is thus also optimal for the constrained problem. Indeed, the closed loop state satisfies  $|x(t)| \leq 1$ .

2. (a) The sequential optimization problem has the form

$$\max \sum_{k=0}^{N-1} u_k \quad \text{subj. to} \quad \begin{cases} x_{k+1} = x_k + \theta(x_k - u_k), & x_0 \text{ given} \\ 0 \leq u_k \leq x_k \end{cases}$$

The corresponding dynamic programming recursion is

$$V_k(x) = \max_{0 \leq u \leq x} \{u + V_{k+1}(x + \theta(x - u))\}$$

$$V_N(x) = 0$$

(b) We have

$$V_3(x) = 0$$

$$V_2(x) = \max_{0 \leq u \leq x} u = x, \quad \Rightarrow u_2 = x$$

$$\begin{aligned} V_1(x) &= \max_{0 \leq u \leq x} \{u + x + \theta(x - u)\} \\ &= \max\{(1 + \theta)x, 2x\} = 2x, \quad \Rightarrow u_1 = x \end{aligned}$$

$$\begin{aligned} V_0(x) &= \max_{0 \leq u \leq x} \{u + 2(x + \theta(x - u))\} \\ &= \max\{(2 + \theta)x, 3x\} = 3x, \quad \Rightarrow u_0 = x \end{aligned}$$

Hence it is optimal to spend all the time. Note, the problem becomes more interesting if the time horizon is longer, i.e. when  $N$  is larger.

3. (a) Let us consider the problem in (a) and (b) simultaneously. The ARE becomes

$$2ap - p^2 = 0$$

We are generally looking for a positive definite solution. We have

$$p = \begin{cases} 2a, & a > 0 \\ 0, & a < 0 \end{cases}$$

The optimal control becomes

$$u = -px = \begin{cases} -2a, & a > 0 \\ 0, & a < 0 \end{cases}$$

The closed loop system  $\dot{x} = (a - p)x = -|a|x$  is stable in both cases. Note that  $p$  only is positive semi-definite when  $a < 0$ , which is a case not covered by the result in the course. However, it is obvious that  $u = 0$  is optimal because we only penalize the control and the system converges to zero with  $u = 0$ .

- (b) The open loop system is unstable when  $a > 0$  and the control  $u = -2ax$  stabilizes the system. The open loop system is stable when  $a < 0$  and no control is needed to bring the state to zero. Since the cost function only penalizes the control it is optimal to do nothing.

Note, the case  $a = 0$  is not well defined. The solution  $u = -\epsilon x$  stabilizes the system but the corresponding cost is

$$\int_0^\infty \epsilon^2 e^{-2\epsilon t} x_0^2 dt = \frac{\epsilon x_0^2}{2}$$

which becomes smaller the smaller  $\epsilon$  is. However, the limit when  $\epsilon = 0$  is not stabilizing, i.e.  $u = 0$  does not stabilize the system.

4. The optimal control problem has the formulation

$$\max x_2(1) \quad \text{subj. to} \quad \begin{cases} \dot{x}_1(t) = -x_1(t) + u(t), & x_1(0) = 0 \\ \dot{x}_2(t) = x_1(t), & x_2(0) = 0 \\ 0 \leq u \leq 1 \end{cases}$$

The Hamiltonian becomes

$$H(x, u, \lambda) = \lambda_1(-x_1 + u) + \lambda_2 x_1$$

From the pointwise optimization we get

$$u = \operatorname{argmax}_{0 \leq u \leq 1} H(x, u, \lambda) = \begin{cases} 1, & \lambda_1 > 0 \\ 0, & \lambda_1 < 0 \end{cases}$$

We thus expect a switching control law. The adjoint equation becomes

$$\begin{aligned} \dot{\lambda}_1 &= \lambda_1 - \lambda_2 \\ \dot{\lambda}_2 &= 0 \end{aligned}$$

with terminal condition determined by

$$\lambda(1) - \nabla\Phi(x(1)) \perp S_f$$

where  $\Phi(x) = x_2$  and  $S_f = \{x : x_1(1) = 0.5\}$ . Hence, we get  $\lambda_1(1) = \text{free}$  and  $\lambda_2(1) = 1$ . We can now solve the adjoint system, which gives

$$\begin{aligned}\lambda_1(t) &= 1 + (\lambda_1(0) - 1)e^t \\ \lambda_2(t) &= 1\end{aligned}$$

There can be at most one switch in the control function since  $\lambda_1(t)$  is a monotonic function. From the problem it is now clear that the control must have the form

$$u(t) = \begin{cases} 1, & 0 \leq t < t_s \\ 0, & t_s < t \leq 1 \end{cases}$$

We can determine the switching time from the constraint  $x(1) = 0.5$ . We have  $x_1(t_s) = 1 - e^{-t_s}$  and

$$x(1) = e^{-(1-t_s)}(1 - e^{-t_s}) = 0.5$$

which gives  $t_s = \ln(\frac{2+e}{2})$ .

5. The optimal control problem is

$$\max \int_0^{t_f} e^{-\alpha t} u(t) p(u(t)) dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = -u(t), & x(0) = x_0 \\ t_f \geq 0, x(t_f) = 0 \end{cases}$$

The Hamiltonian is  $H(t, x, u, \lambda) = e^{-\alpha t} u p(u) - \lambda u$ , and the terminal manifold is  $S_f(t) = \{0\}$ . The adjoint equation is  $\dot{\lambda} = -H_x = 0$ , which implies  $\lambda(t) = \text{const}$  (the terminal condition gives no information). The pointwise optimization gives

$$u = \operatorname{argmax}_u e^{-\alpha t} u p(u) - \lambda u = \operatorname{argmax}_u e^{-\alpha t} u(1 - u/2) - \lambda u = 1 - \lambda e^{\alpha t}$$

where we used that  $u \leq 2$  because the cost is zero for  $u > 2$ , which obviously cannot be optimal. From PMP we have the condition (along the optimal solution)

$$H(t_f, x(t_f), u(t_f), \lambda(t_f)) = - \sum_{k=1}^p \nu_k \frac{\partial g_k}{\partial t}(t_f, x(t_f)) - \frac{\partial \phi}{\partial t}(t_f, x(t_f)) = 0$$

$$\Leftrightarrow e^{-\alpha t_f} u(t_f) p(u(t_f)) - \lambda u(t_f) = 0$$

which implies  $u(t_f) = 0$ . Since  $u(t_f) = 1 - \lambda e^{\alpha t_f} = 0$  we must have  $\lambda = e^{-\alpha t_f}$ . The optimal control is  $u(t) = 1 - e^{-\alpha(t_f - t)}$ , where  $t_f$  is determined by the state constraint

$$x(t_f) = x_0 - \int_0^{t_f} u(t) dt = x_0 - t_f - \frac{1}{\alpha}(1 - e^{-\alpha t_f}) = 0 \quad (1)$$

which is a nonlinear equation in  $t_f$ . The optimal value function can now be computed

$$\begin{aligned} V(x, t_f) &= \int_0^{t_f} (u - u^2/2)e^{-\alpha t} dt = 0.5 \int_0^{t_f} (1 - e^{2\alpha(t-t_f)})e^{-\alpha t} dt \\ &= \frac{(1 - e^{-\alpha t_f})^2}{2\alpha} \end{aligned}$$

where once again the terminal time  $t_f$  is implicitly defined by equation (1).