5,6:

Linear Algebra Interpretations

$$A = \begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad \mathcal{F} = \{ x \in \mathbb{R}^5 : Ax = b \}.$$

$$x^{(1)} \in \mathcal{F},\ x^{(2)} \in \mathcal{F} \Rightarrow A(x^{(1)} - x^{(2)}) = Ax^{(1)} - Ax^{(2)} = b - b = 0$$

i.e. $x^{(1)} - x^{(2)} \in \mathcal{N}(A)$.

Consider basic solution corresponding to $\beta = \{1,3\}$ $\nu = \{2,5,4\}$.

Let $\tilde{A} = [A_\beta \ A_\nu] = [a_1 \ a_3 \ a_2 \ a_5 \ a_4] = \begin{bmatrix} 4 & 2 & 3 & 0 & 1 \\ 0 & 2 & 1 & 4 & 3 \end{bmatrix}$

Apply Gauss-Jordan

$$\tilde{A} \sim \begin{bmatrix} 1 & 1/2 & 3/4 & 0 & 1/4 \\ 0 & 1 & 1/2 & 2 & 3/2 \end{bmatrix} \sim \Bigg[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{=I} \ \underbrace{\begin{matrix} \boxed{\begin{matrix} 1/2 \\ 1/2 \end{matrix}} & \begin{matrix} -1 & -1/2 \\ 2 & 3/2 \end{matrix} \end{matrix}}_{= A_\beta^{-1} A_\nu} \Bigg] = U$$

$= \bar{a}_2$ in 5.6 b.

since A_β^{-1} exists

Note: $\tilde{A} = [A_\beta \ A_\nu] = A_\beta [I \ \ A_\beta^{-1} A_\nu] = A_\beta \underbrace{[I \ \ \bar{a}_{\nu_1} \ \bar{a}_{\nu_2} \ \bar{a}_{\nu_3}]}_{=U}$

Know: $\tilde{A}x = A_\beta [I \ \ \bar{a}_{\nu_1} \ \bar{a}_{\nu_2} \ a_{\nu_3}] \begin{bmatrix} x_\beta \\ x_\nu \end{bmatrix} = b$

i.e. $x_\beta + \bar{a}_{\nu_1} x_{\nu_1} + \bar{a}_{\nu_2} x_{\nu_2} + a_{\nu_3} x_{\nu_3} = A_\beta^{-1} b = \bar{b}$

$$\mathcal{N}(\tilde{A}) = \mathcal{N}(U) = \left\{ I \cdot \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/2 & -1 & -1/2 \\ 1/2 & 2 & 3/2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \\ x_4 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} +1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} x_5 + \begin{bmatrix} +1/2 \\ 0 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} x_4 \right\} = \text{Span} \left\{ \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

How x changes if x_2 increases.

How x changes if x_5 increases

How x changes if x_4 increases.

x_2 decrease makes $x_2 \leq 0$

5.8: Linear Algebra Interpretations

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad c^T = [-1\ \ -1\ \ -2\ \ 0\ \ 0]$$

$$\mathcal{B} = \{3,5\} \qquad \mathcal{V} = \{1,2,4\} \qquad A_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad A_{\mathcal{V}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} A_{\mathcal{B}} & A_{\mathcal{V}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix} = U$$

$$\mathcal{N}(\tilde{A}) = \mathcal{N}(U) = \left\{ I \cdot \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -2 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ +1 \\ 0 \\ -0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} x_4 \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

When x_2 increases no other element decreases.

Furthermore, how does the objective value vary with x_2?

$$c^T \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 = c_2 x_2 - c_{\mathcal{B}}^T \begin{bmatrix} -1 \\ 0 \end{bmatrix} x_2 = -3 x_2 = r_2 x_2$$

$[c_3\ \ c_5] = [-2\ \ 0]$

So the objective value goes to $-\infty$ when $x_2 \to \infty$.

The reduced cost for non-basic variable x_2

Note: $c^T \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -2 \end{bmatrix} = 1 = r_1 \qquad c^T \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 2 = r_4$

The vector d asked for is $d = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$