EXAM FOR OPTIMIZATION SF1811/SF1831/SF1841

JUNE 8, 2011

Examiner: Amol Sasane (Phone: 790 7320)
Writing time allowed: 1400-1900
Writing material: Pen or pencil, eraser and a ruler is allowed. Calculators are **not** allowed! A formula sheet is provided.
Instructions: Motivate your answers carefully. Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets. A passing grade is guaranteed for 25 points (including bonus points from the voluntary home assignments).

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(1) (a) Consider the linear programming problem

$$(\mathrm{P}) : \left\{ \begin{array}{rl} \text{minimize} & x_1 + 2x_2 + 3x_3 + \cdots + 2011x_{2011} \\ \text{subject to} & x_1 \geq 1 \\ & x_1 + x_2 \geq 2 \\ & x_1 + x_2 + x_3 \geq 3 \\ & \vdots \\ & x_1 + x_2 + x_3 + \cdots + x_{2011} \geq 2011 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0, \dots,\ x_{2011} \geq 0. \end{array} \right.$$

Check that the vector

$$\widehat{x} = 2011e_1 := \left[\begin{array}{cccc} 2011 & 0 & \dots & 0 \end{array}\right]^\top \in \mathbb{R}^{2011}$$

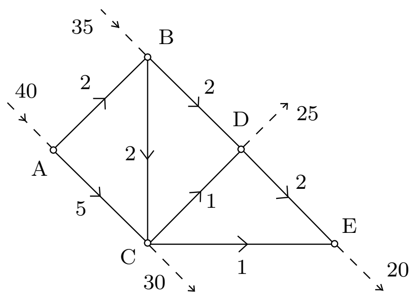
is feasible for (P).
Determine the dual linear programming problem (D) to the primal problem (P).
Check that the vector

$$\widehat{y} = e_{2011} := \left[\begin{array}{cccc} 0 & \dots & 0 & 1 \end{array}\right]^\top \in \mathbb{R}^{2011}$$

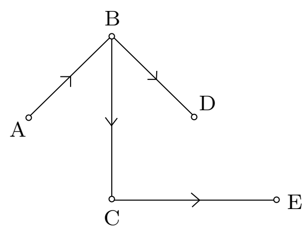
is feasible for (D).
Using duality theory, explain why $\widehat{x}$ is an optimal solution for (P).

(5 points)

(b) Consider the minimum cost of flow problem for the network shown below.



We have labelled the 5 nodes as A,B,C,D,E. The nodes A and B are source nodes with a supply of 40 and 35 units, respectively, while the nodes C, D and E are sink nodes with demands of 30, 25 and 20 units, respectively. Beside each directed edge we have indicated the cost c_{ij} per unit flow.
Show that the basic feasible solution corresponding to the spanning tree given below is an optimal solution for the minimum network flow problem associated with this network.



(5 points)

(2) (a) Find a basis for the range and the kernel of the matrix

$$A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}.$$

(5 points)

(b) Consider the following quadratic optimization problem:

$$\begin{cases} \text{minimize} & x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1 \\ \text{such that} & x_1 + 2x_2 + 3x_3 = 6 \\ & 3x_1 + 2x_2 + x_3 = 6. \end{cases}$$

Using the Lagrange method, verify that $\widehat{x} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ is an optimal solution.

(5 points)

(3) (a) Consider the linear programming problem given by

$$\begin{cases} \text{maximize} & x_1 + 2x_2 - 2x_3 \\ \text{such that} & x_1 \qquad + 3x_3 \leq 1 \\ & x_1 + x_2 - x_3 \leq 1 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{cases}$$

(Note that it is a maximization problem.)
Write the problem down as a linear programming problem in the standard form.
Solve it using the simplex method, by starting with the slack variables as the basic variables.

(5 points)

(b) Use the method of Lagrange multipliers to find an optimal solution to the following problem:

$$\begin{cases} \text{minimize} & x_1 - x_2 \\ \text{such that} & x_1^2 + x_2^2 - 2x_2 = 0. \end{cases}$$

(5 points)

(4) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x) = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_3 + 4x_2x_3 + x_3^4 + x_1 \quad (x \in \mathbb{R}^3).$$

(a) Is f a convex function on $\mathbb{R}^3$?

(3 points)

(b) Consider the uncontrained optimization problem

$$\begin{cases} \text{minimize} & f(x) \\ \text{such that} & x \in \mathbb{R}^3. \end{cases}$$

Starting with the point $x^{(1)} = 0 \in \mathbb{R}^3$, use Newton's method to find the next iteration point $x^{(2)}$.

(You may use without justification $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 4 & 10 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.)

(2 points)

(c) Now consider the constrained optimization problem given by

$$\begin{cases} \text{minimize} & f(x) \\ \text{such that} & 0 \leq x_1 \leq 1, \\ & 0 \leq x_2 \leq 1, \\ & 0 \leq x_3 \leq 1. \end{cases}$$

Does the point $\widehat{x} := 0 \in \mathbb{R}^3$ satisfy the KKT-conditions for this problem?
Can we conclude that $\widehat{x}$ is an optimal solution? Justify your answer.

(5 points)

(5) (a) Consider the problem

$$(\mathrm{P}): \begin{cases} \text{minimize} & x_1^3 + x_2^3 \\ \text{such that} & 4x_1 + 9x_2 \geq 1, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{cases}$$

(i) Write down the Lagrange relaxed problems associated with the above problem by relaxing the inequality constraint $4x_1 + 9x_2 \geq 1$.

(ii) Determine the dual problem (D) to (P) and find an optimal solution $\widehat{y}$ for (D).

(iii) Find an optimal solution $\widehat{x}$ to the problem (P). Justify your answer.

(7 points)

(b) Determine if the following statements are true or false. All the statements below refer to optimization problems of the form:

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{F}, \end{cases}$$

where the variable x takes values in the subset $\mathcal{F}$ of $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function on $\mathbb{R}^n$.

(i) Whenever the problem is convex, the Hessian of f is positive semidefinite at all points of $\mathcal{F}$.

(ii) Whenever the problem is convex, it must have at least one optimal solution.

(iii) Whenever the problem is convex, it has at most one optimal solution.

(3 points)

End of the exam.