Solution to the exam for SF1811/SF1831/SF1841

March 20, 2010

(1) The problem is in standard form

$$(LP):\begin{cases}\text{minimize} & c^\top x\\ \text{subject to} & Ax=b,\\ & x\geq 0,\end{cases}$$

where

$$A=\begin{bmatrix}3 & -3 & 1 & 0\\ 6 & -2 & 0 & 1\end{bmatrix},\quad b=\begin{bmatrix}3\\2\end{bmatrix},\quad c=\begin{bmatrix}4\\1\\-1\\2\end{bmatrix}.$$

We start with x_3 and x_4 as basic variables. Thus $\beta=(3,4)$ and $\nu=(1,2)$. Then

$$A_\beta=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\text{ and }A_\nu=\begin{bmatrix}3 & -3\\ 6 & -2\end{bmatrix}.$$

The initial basic solution is $x_\beta=\overline{b}$, where $A_\beta\overline{b}=b$, that is,

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\overline{b}=\begin{bmatrix}3\\2\end{bmatrix},\text{ and so }\overline{b}=\begin{bmatrix}3\\2\end{bmatrix}.$$

The simplex multipliers vector y is obtained by solving $A_\beta^\top y=c_\beta$, that is,

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}y=\begin{bmatrix}-1\\2\end{bmatrix},\text{ and so }y=\begin{bmatrix}-1\\2\end{bmatrix}.$$

The reduced costs of the nonbasic variables are given by $r_\nu=c_\nu-A_\nu^\top y$, that is,

$$r_\nu=\begin{bmatrix}4\\1\end{bmatrix}-\begin{bmatrix}3 & 6\\ -3 & -2\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}-\begin{bmatrix}9\\-1\end{bmatrix}=\begin{bmatrix}-5\\2\end{bmatrix}.$$

Since $r_{\nu_1}=r_1=-5<0$ and it is the smallest, we make x_1 a new basic variable. We compute $\overline{a}_1$ using $A_\beta\overline{a}_1=a_1$, that is,

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\overline{a}_1=\begin{bmatrix}3\\6\end{bmatrix},\text{ and so }\overline{a}_1=\begin{bmatrix}3\\6\end{bmatrix}.$$

Then the new basic variable x_1 can increase up to

$$t_{max}=\min\left\{\frac{\overline{b}_k}{\overline{a}_{1,k}}:\overline{a}_{1,k}>0\right\}=\min\left\{\frac{3}{3},\frac{2}{6}\right\}=\frac{1}{3}=\frac{\overline{b}_2}{\overline{a}_{1,2}}.$$

The minimizing index is $k=2$, and hence $x_{\beta_2}=x_4$ leaves the set of basic variables, and x_1 takes its place. So $\beta=(3,1)$ and $\nu=(4,2)$. Hence

$$A_\beta=\begin{bmatrix}1 & 3\\ 0 & 6\end{bmatrix}\text{ and }A_\nu=\begin{bmatrix}0 & -3\\ 1 & -2\end{bmatrix}.$$

We calculate $\overline{b}$ using $A_\beta \overline{b} = b$, that is,

$$\begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix} \overline{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \text{ and so } \overline{b} = \begin{bmatrix} 2 \\ \frac{1}{3} \end{bmatrix}.$$

The simplex multipliers vector y is obtained by solving $A_\beta^\top y = c_\beta$, that is,

$$\begin{bmatrix} 1 & 0 \\ 3 & 6 \end{bmatrix} y = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \text{ and so } y = \begin{bmatrix} -1 \\ \frac{7}{6} \end{bmatrix}.$$

The reduced costs of the nonbasic variables are given by $r_\nu = c_\nu - A_\nu^\top y$, that is,

$$r_\nu = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{7}{6} \end{bmatrix} = \begin{bmatrix} 2 - \frac{7}{6} \\ 1 - 3 + \frac{7}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{1}{3} \end{bmatrix}.$$

Since $r_\nu \geq 0$, the current basic feasible solution is optimal. So

$$\widehat{x} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

is optimal for (LP).

(2) The dual problem is

$$(D): \begin{cases} \text{maximize} & -c^\top y \\ \text{subject to} & A^\top y \leq c, \\ & y \geq 0. \end{cases}$$

But since $A^\top = -A$, the dual problem becomes

$$(D): \begin{cases} \text{maximize} & -c^\top y \\ \text{subject to} & -Ay \leq c, \\ & y \geq 0. \end{cases}$$

This can be rewritten as

$$(D): \begin{cases} \text{minimize} & c^\top y \\ \text{subject to} & Ay \geq -c, \\ & y \geq 0, \end{cases}$$

which is the same as the primal problem. Since $\mathcal{F}_P \neq \emptyset$ (given), from the above we obtain $\mathcal{F}_D = \mathcal{F}_P \neq \emptyset$ as well. So by the Duality Theorem, both the primal problem (P) as well as the dual problem (D) have optimal solutions. Also since (P)=(D), the set of their optimal solutions is the same. Hence if $\widehat{x}$ is optimal for (P), then it is optimal for (D). From the Duality Theorem, we then obtain that $c^\top \widehat{x} = -c^\top \widehat{x}$ and so $c^\top \widehat{x} = 0$. So the optimal value of (P) is zero.

(3)(a) Call F1 source 1, F2 as source 2 and F3 as source 3. Call A, B, C, D as destination 1, 2, 3, 4, respectively. For i =1, 2, 3 and j =1, 2, 3, 4, let

$x_{ij} =$ amount in tonnes that F transports from source i to destination j,
$d_j =$ capacity in tonnes of destination j,
$s_i =$ amount in tonnes of product P produced at source i,
$c_{ij} =$ cost of transportation in units of 1000 SEK/tonnes of product from source i to destination j.

Then

$$\begin{array}{l} s_1 = 250 \\ s_2 = 250 \\ s_3 = 500 \end{array} \quad \text{and} \quad \begin{array}{l} d_1 = 150 \\ d_2 = 200 \\ d_3 = 300 \\ d_4 = 350, \end{array}$$

while the c_{ij} are given by

c_{ij}	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	10	5	11	11
$i = 2$	10	2	7	12
$i = 3$	9	1	4	8

We note that

$$s_1 + s_2 + s_3 = 250 + 250 + 500 = 1000,$$

and

$$d_1 + d_2 + d_3 + d_4 = 150 + 200 + 300 + 350 = 1000$$

as well. So the problem can be formulated as the following balanced transportation problem:

$$(TP): \begin{cases} \text{minimize} & \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} \\ \text{subject to} & \displaystyle\sum_{j=1}^{4} x_{ij} = s_i \text{ for } i = 1, 2, 3, \\ & \displaystyle -\sum_{i=1}^{3} x_{ij} = -d_j \text{ for } j = 1, 2, 3, 4, \\ & x_{ij} \geq 0 \text{ for all } i, j. \end{cases}$$

(3)(b) We find the following basic feasible solution using the northwest corner method:

x_{ij}	$j = 1$	$j = 2$	$j = 3$	$j = 4$	s_i
$i = 1$	150	100	–	–	250
$i = 2$	–	100	150	–	250
$i = 3$	–	–	150	350	500
d_j	150	200	300	350	

(3)(c) The simplex multipliers u_i and v_j corresponding to the above basic feasible solution can be found using the relation $c_{ij} = u_i - v_j$ for basic variable indices and with $v_4 = 0$:

c_{ij}	$j = 1$	$j = 2$	$j = 3$	$j = 4$	u_i
$i = 1$	10	5			14
$i = 2$		2	7		11
$i = 3$			4	8	8
v_j	4	9	4	0	

The reduced costs r_{ij} for the nonbasic variables can be found out using $r_{ij} = c_{ij} - u_i + v_j$:

r_{ij} (c_{ij})	$j = 1$	$j = 2$	$j = 3$	$j = 4$	u_i
$i = 1$	–	–	1 (11)	−3 (11)	14
$i = 2$	3 (10)	–	–	1 (12)	11
$i = 3$	5 (9)	2 (1)	–	–	8
v_j	4	9	4	0	

Since $r_{14} = -3 < 0$, this basic feasible solution is not optimal. Let x_{14} be a new basic variable. Set $x_{14} = t$ and let t increase from 0, while the other nonbasic variables stay at 0. Then we have:

$x_{ij}(t)$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	s_i
$i = 1$	150	$100 - t$	–	t	250
$i = 2$	–	$100 + t$	$150 - t$	–	250
$i = 3$	–	–	$150 + t$	$350 - t$	500
d_j	150	200	300	350	

We see that t can increase up to 100, and $x_{12}(t)|_{t=100} = 0$. So the new basic feasible solution is:

x_{ij}	$j = 1$	$j = 2$	$j = 3$	$j = 4$	s_i
$i = 1$	150	–	–	100	250
$i = 2$	–	200	50	–	250
$i = 3$	–	–	250	250	500
d_j	150	200	300	350	

The new simplex multipliers are:

c_{ij}	$j = 1$	$j = 2$	$j = 3$	$j = 4$	u_i
$i = 1$	10			11	11
$i = 2$		2	7		11
$i = 3$			4	8	8
v_j	1	9	4	0	

The reduced costs are:

r_{ij} (c_{ij})	$j=1$	$j=2$	$j=3$	$j=4$	u_i
$i=1$	–	3 (5)	4 (11)	–	11
$i=2$	0 (10)	–	–	1 (12)	11
$i=3$	2 (9)	2 (1)	–	–	8
v_j	1	9	4	0	

Since all the $r_{ij} \geq 0$, this solution is optimal.

(3)(d) If $\widetilde{c}_{i1} = c_{i1} + 2$, then we note that the feasible set is the same, while the cost corresponding to a feasible solution is now given by

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=2}^{4} c_{ij}x_{ij} + \sum_{i=1}^{3}\widetilde{c}_{i1}x_{i1} &= \sum_{i=1}^{3}\sum_{j=2}^{4} c_{ij}x_{ij} + \sum_{i=1}^{3}(c_{i1}+2)x_{i1} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} + 2\sum_{i=1}^{3} x_{i1} = \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} + 2d_1 \\
&= \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} + 2 \cdot 150.
\end{aligned}$$

Hence the optimal solution is the same.

(4) Let h_1, h_2, h_3 be the true heights of the hills above sea level. The errors $e_1, \dots, e_6$ in the six measurements are then given as follows:

$$\begin{aligned}
e_1 &= h_1 - 1236 \\
e_2 &= h_2 - 1941 \\
e_3 &= h_3 - 2417 \\
e_4 &= h_2 - h_1 - 711 \\
e_5 &= h_3 - h_1 - 1177 \\
e_6 &= h_3 - h_2 - 474.
\end{aligned}$$

The problem is to minimize $e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2$, that is,

$$\begin{cases} \text{minimize} & \|Ax - b\|^2 \\ \text{subject to} & x \in \mathbb{R}^3, \end{cases}$$

where

$$A = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{array}\right], \quad b = \begin{bmatrix} 1236 \\ 1941 \\ 2417 \\ 711 \\ 1177 \\ 474 \end{bmatrix} \text{ and } x = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}.$$

$\widehat{x}$ is optimal iff it satisfies the normal equation $A^\top A\widehat{x} = A^\top b$. We have

$$A^\top A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \text{ and } A^\top b = \begin{bmatrix} -652 \\ 2178 \\ 4068 \end{bmatrix}.$$

Thus adding all equations in the system $A^\top A\widehat{x} = A^\top b$ gives us that

$$\widehat{x}_1 + \widehat{x}_2 + \widehat{x}_3 = 4068 + 2178 - 652 = 5594.$$

Now adding the equation $\widehat{x}_1 + \widehat{x}_2 + \widehat{x}_3 = 5594$ to each of the equations in the system $A^\top A\widehat{x} = A^\top b$ yields

$$\begin{aligned} \widehat{x_1} &= \frac{-652 + 5594}{4} = 1235.5, \\ \widehat{x_2} &= \frac{2178 + 5594}{4} = 1943, \\ \widehat{x_3} &= \frac{4068 + 5594}{4} = 2415.5. \end{aligned}$$

So upon minimizing the least squares error associated with the measurements, the estimated heights of the hills H_1, H_2, H_3 are 1235.5, 1943, 2415.5 meters, respectively.

(5) The problem can be rewritten as follows:

$$\left\{ \begin{array}{rl} \text{minimize} & f(x) := -x_5 \\ \text{subject to} & h_1(x) := x_1 + x_2 + x_3 + x_4 + x_5 - 8 = 0, \\ & h_2(x) := x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 16 = 0, \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 \in \mathbb{R}. \end{array} \right.$$

We have

$$\begin{aligned} \nabla h_1(x) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \\ \nabla h_2(x) &= \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 & 2x_4 & 2x_5 \end{bmatrix}. \end{aligned}$$

Suppose α and β are scalars, not both zeros, such that

$$\alpha \nabla h_1(x) + \beta \nabla h_2(x) = 0.$$

Since $\nabla h_1(x) \neq 0$, it follows that $\beta \neq 0$, and so $\nabla h_2(x) = k \nabla h_1(x)$ for some scalar k. Hence $x_1 = \cdots = x_5$. But then $h_1(x) = 0$ gives

$$x_1 = \cdots = x_5 = \frac{8}{5},$$

and then $h_2(x) = 5 \cdot \frac{64}{25} - 16 \neq 0$. So $\nabla h_1(x)$ and $\nabla h_2(x)$ are independent for every feasible x, and so every feasible x is a regular point.

Thus if x is a local optimal solution, then there exists a

$$u = \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \in \mathbb{R}^2$$

such that $\nabla f(x) + u^\top \nabla h(x) = 0$, that is,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} \lambda & \mu \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2x_1 & 2x_2 & 2x_3 & 2x_4 & 2x_5 \end{bmatrix} = 0.$$

Hence

$$\begin{aligned} \lambda + 2\mu x_1 &= 0, & (1)\\ \lambda + 2\mu x_2 &= 0, & (2)\\ \lambda + 2\mu x_3 &= 0, & (3)\\ \lambda + 2\mu x_4 &= 0, & (4)\\ -1 + \lambda + 2\mu x_5 &= 0. & (5) \end{aligned}$$

We consider the two cases $\lambda = 0$ and $\lambda \neq 0$ separately.

1° If $\lambda = 0$, then (5) gives $2\mu x_5 = 1$ and so $\mu \neq 0$. But then (1)-(4) give $x_1 = x_2 = x_3 = x_4 = 0$. So $h_1(x) = 0$ now gives $x_5 = 8$. But then $h_2(x) = 64 - 16 \neq 0$. So this case gives no feasible x.

2° Suppose $\lambda \neq 0$. The (1) gives $2\mu x_1 = -\lambda$ and so $\mu \neq 0$. Then (1)-(4) give $x_1 = x_2 = x_3 = x_4 = -\frac{\lambda}{2\mu} = k$ (say). Then $h_1(x) = 0$ gives $4k + x_5 - 8 = 0$, while $h_2(x) = 0$ gives $4k^2 + x_5^2 - 16 = 0$. Eliminating k, we obtain

$$x_5^2 + 4\left(\frac{8 - x_5}{4}\right)^2 - 16 = 0,$$

and upon simplifying, we obtain $x_5(\frac{5}{4}x_5 - 4) = 0$. Thus $x_5 = \frac{16}{5}$ or $x_5 = 0$. Hence $x = (\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{16}{5})$ or $x = (2, 2, 2, 2, 0)$. Both of these are feasible, and since $\frac{16}{5} > 0$, we conclude that if there is an optimal solution, it must be $x = (\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{16}{5})$.

The feasible set $\mathcal{F}$, namely

$$\{x \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 8\} \cap \{x \in \mathbb{R}^5 : x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 16\}$$

is bounded (indeed, $\mathcal{F}$ is contained in the ball with center 0 and radius 4), and it is also closed (since it is the intersection of two closed sets). So $\mathcal{F}$ is compact. The map $x \mapsto -x_5$ is continuous. So we know that $f : \mathcal{F} \to \mathbb{R}$ has a global minimum on $\mathcal{F}$. Consequently, $x = (\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{16}{5})$ is a global minimizer.

So the largest value of x_5 is $\frac{16}{5}$.

(6)(a) $x \overset{\varphi}{\mapsto} -(x_1 + x_2)$ is convex. The map $y \overset{\psi}{\mapsto} e^y$ is increasing. Thus $x \overset{\psi\circ\varphi}{\mapsto} e^{-(x_1+x_2)}$ is convex. Also since $x \mapsto e^{x_1}$ and $x \mapsto e^{x_2}$ are convex, so is $x \mapsto e^{x_1} + e^{x_2} - 20$. Finally, $x \mapsto -x_1$ is convex. Thus f, g_1, g_2 defined by

$$\begin{aligned} f(x) &= e^{-(x_1+x_2)},\\ g_1(x) &= e^{x_1} + e^{x_2} - 20,\\ g_2(x) &= -x_1, \end{aligned}$$

are all convex. Hence the given problem

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g_1(x) \leq 0, \\ & g_2(x) \leq 0 \end{cases}$$

is a convex optimization problem. Also $g_1(1,0) = e^1 + 1 - 20 < 0$ and $g_2(1,0) = -1 < 0$. So the problem is regular as well.

(6)(b) For a regular convex problem, x is optimal iff the KKT-conditions hold, that is, that there exists a $y \in \mathbb{R}^2$ such that the following hold:

(KKT-1) $\nabla f(x) + y^\top g(x) = 0$ that is,

$$\begin{bmatrix} -e^{-(x_1+x_2)} & -e^{-(x_1+x_2)} \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} e^{x_1} & e^{x_2} \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

So $e^{-(x_1+x_2)} - y_1 e^{x_1} + y_2 = 0$ and $e^{-(x_1+x_2)} - y_1 e^{x_2} = 0$.

(KKT-2) $g_i(x) \leq 0$ for all i, that is, $x_1 \geq 0$ and $e^{x_1} + e^{x_2} \leq 20$.

(KKT-3) $y \geq 0$, that is, $y_1 \geq 0$ and $y_2 \geq 0$.

(KKT-4) $y_i g_i(x) = 0$ for all i, that is, $y_1(e^{x_1} + e^{x_2} - 20) = 0$ and $y_2 x_1 = 0$.

If $x_1 = 0$, then (KKT-1) gives $y_1 = e^{-2x_2} \neq 0$. (KKT-4) then gives that $e^{x_1} + e^{x_2} - 20$ must be 0, and since $x_1 = 0$, we further obtain that $e^{x_2} = 19$. (KKT-1) gives $y_2 = e^{-2x_2} - e^{-x_2} = \frac{1}{19}(\frac{1}{19} - 1) < 0$, contradicting (KKT-3). So it cannot be the case that $x_1 = 0$.

If $x_1 \neq 0$, then (KKT-4) gives $y_2 = 0$. (KKT-1) then gives first of all that $y_1 = e^{-x_1 - 2x_2} > 0$. Also, $y_1(e^{x_1} - e^{x_2}) = 0$ and since $y_1 > 0$, we obtain $e^{x_1} = e^{x_2}$, which implies that $x_1 = x_2$. (KKT-4) together with $y_1 > 0$ gives $e^{x_1} + e^{x_2} - 20 = 0$. Since $x_1 = x_2$ we now obtain that $e^{x_1} = e^{x_2} = 10$, so that $x_1 = x_2 = \log_e 10$. Then it is easily verified that (KKT-1) to (KKT-4) hold with $x_1 = x_2 = \log_e 10$, $y_1 = e^{-x_1 - 2x_2} = e^{-3\log_e 10} = \frac{1}{1000}$ and $y_2 = 0$. So the global optimal solution is given by $x_1 = x_2 = \log_e 10$.

(7) Let $X = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0\}$. Define $L : X \times \mathbb{R} \to \mathbb{R}$ by

$$L(x,y) = x_1 + x_2 + y\left(\frac{1}{x_1} + \frac{1}{x_2} - 1\right) \quad (x \in X,\ y \in \mathbb{R}).$$

The relaxed Lagrange problem (PR_y) is the following:
Given $y \geq 0$, minimize $x \mapsto L(x,y)$ on X, that is,

$$(PR_y) : \begin{cases} \text{minimize} & x_1 + x_2 + y\left(\dfrac{1}{x_1} + \dfrac{1}{x_2} - 1\right) \\ \text{subject to} & x_1 > 0 \text{ and } x_2 > 0. \end{cases}$$

For nonnegative a, b, we have $\frac{a+b}{2} \geq \sqrt{ab}$ with equality iff $a = b$. Hence

$$\begin{aligned} x_1 + x_2 + y\left(\frac{1}{x_1} + \frac{1}{x_2} - 1\right) &= x_1 + \frac{y}{x_1} + x_2 + \frac{y}{x_2} \geq 2\sqrt{x_1 \frac{y}{x_1}} + 2\sqrt{x_2 \frac{y}{x_2}} \\ &= 2\sqrt{y} + 2\sqrt{y} = 4\sqrt{y}, \end{aligned}$$

with equality iff $x_1 = \frac{y}{x_1}$ and $x_2 = \frac{y}{x_2}$, that is, iff $x_1 = \sqrt{y}$ and $x_2 = \sqrt{y}$. So $x_1 = \widehat{x}_1(y) = \sqrt{y}$ and $x_2 = \widehat{x}_2(y) = \sqrt{y}$. The dual objective function is

$$\varphi(y) = L(\widehat{x}(y), y) = \sqrt{y} + \sqrt{y} + y\left(\frac{1}{\sqrt{y}} + \frac{1}{\sqrt{y}} - 1\right) = 4\sqrt{y} - y.$$

We seek a maximum over $y \geq 0$. We have $\varphi'(y) = \frac{4}{2\sqrt{y}} - 1 = 0$ iff $y = 4$. We see that $\varphi(y) = \sqrt{y}(4 - \sqrt{y}) < 0$ for $y > 16$. On the compact interval $[0, 16]$, φ has a maximum. $\varphi(0) = \varphi(16) = 0$ and $\varphi(4) = 4$, $\varphi'(4) = 0$ show that $\widehat{y} = 4 \geq 0$ is indeed a maximizer.

Set $\widehat{x}_1 := \widehat{x}_1(\widehat{y}) = 2$ and $\widehat{x}_2 := \widehat{x}_2(\widehat{y}) = 2$. Then $(\widehat{x}_1, \widehat{x}_2)$ is feasible for the original problem since $\frac{1}{\widehat{x}_1} + \frac{1}{\widehat{x}_2} = \frac{1}{2} + \frac{1}{2} = 1$, $\widehat{x}_1 = 2 > 0$, $\widehat{x}_2 = 2 > 0$. $\widehat{y} = 4 \geq 0$ and $\widehat{y}\left(\frac{1}{\widehat{x}_1} + \frac{1}{\widehat{x}_2} - 1\right) = \widehat{y} \cdot 0 = 0$. Thus $(\widehat{x}, \widehat{y})$ satisfy the global optimality conditions associated with the problem and so $\widehat{x} = (2, 2)$ is an optimal solution.