KTH Mathematics

Exam in SF1811/SF1831/SF1841 Optimization.
Monday June 11, 2012, time: 14.00–19.00

Examiner: Per Enqvist, tel. 790 62 98

Allowed utensils: Pen, paper, eraser and ruler. **No calculator!** A formula-sheet is handed out.

Solution methods: If not specifically stated in the problem statement, the problems should be solved using systematic methods that do not become futile for large problems. Unless so stated, known theorems can be used without proving them, assuming that they are stated correctly. Motivate your conclusions carefully.

A passing grade is guarranteed for 25 points, including bonus points from the voluntary home assignments. 23-24 points grant the possibility to complement the exam within three weeks from the announcement of the results. Contact the instructor asap if this is the case.

Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets. (This is important since the exams are split up for fair correction)

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1. (a) We have two resistors with the unknown resistances R_1 and R_2 that we want to determine using the least squares method.

We have a battery of 12 Volts and we can measure currents. First we connect the first resistor to the battery and measures the current to 0.3 Ampere. Then we connect the second resistor to the battery and measures the current to 0.4 Ampere. Finally, we connect the two resistors to the battery in parallell and measures the current to 0.6 Ampere (before it divides up to the two resistors).

Formulate the linear least-squares problem to determine the two resistances, and solve it. .... (5p)

Hint: it may be useful to use $L_1 = 1/R_1$ and $L_2 = 1/R_2$ as variables.

(b) Let

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 & 2 \\ 4 & 4 & 0 & 1 \end{bmatrix}$$

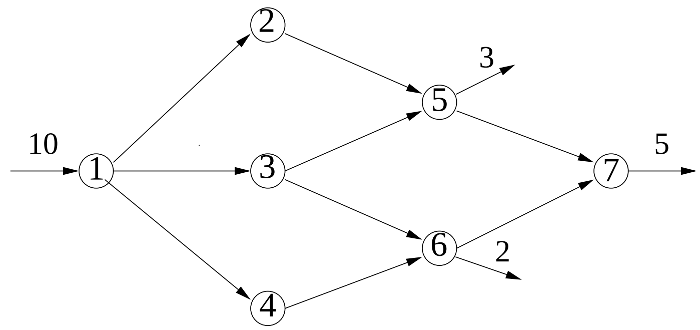
Determine the nullspace and range of $\mathbf{A}$.

Is the vector

$$\mathbf{v} = \begin{bmatrix} 0 & 4 & 2 & 3 \end{bmatrix}^T$$

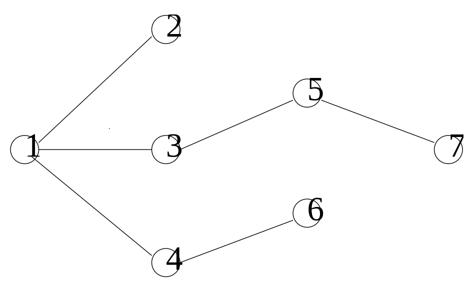
in the range of $\mathbf{A}^T$? .... (4p)

2. Consider the following network problem:



(a) Determine the adjacency matrix for this network and the vector of external flows, i.e., the matrix $\mathbf{A}$ and vector $\mathbf{b}$ such that $\mathbf{Ax} = \mathbf{b}$ makes sure that there is a balanced flow in each node. Let $\mathbf{x} = (x_{12}, x_{13}, x_{14}, x_{25}, x_{35}, x_{36}, x_{46}, x_{57}, x_{67})^T$. ....(3p)

(b) Determine the basic solution that corresponds to the spanning tree in the figure below. Is it feasible? Is it a degenerated basic solution?



....(2p)

(c) Assume that the costs are $c_{12} = 2$, $c_{13} = 3$, $c_{14} = 1$, $c_{25} = 4$, $c_{35} = 2$, $c_{36} = 3$, $c_{46} = 1$, $c_{57} = 3$, $c_{67} = 2$.

Use the network version of the Simplex algorithm to determine if the basic solution in (b) is optimal. If not, determine an optimal solution. ....(5p)

3. (a) Consider the following linear optimization problem:

$$(P) \quad \left[\begin{array}{ll} \min\limits_{x} & 2x_1 - 2x_2 + x_3 + x_4 \\ \text{s.t.} & x_1 + 3x_2 + x_4 = 4 \\ & x_1 + x_2 + x_3 + x_4 = 2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0. \end{array} \right].$$

Solve (P) with the Simplex-algorithm.
Start with the basic variable indecies $\beta = \{1, 2\}$, i.e. let x_1 and x_2 be basic variables. . . . . (4p)

(b) Determine explicitly the dual (D) of (P). (no matrices in the answer) . . . . (2p)

(c) Determine the optimal solution to the dual and show that the complementarity conditions are satisfied for the optimal solutions to the primal and dual problems. . . . . (4p)

4. Consider the quadratic optimization problem

$$(QP) \quad \left[\text{ minimize } \quad f(x) \right],$$

where $f(x) = \frac{1}{2}x^T Hx + c^T x$ and H, c are given by

$$H = \begin{bmatrix} 4 & -4 \\ -4 & 5 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

(a) The optimization problem (QP) can easily be solved, but could also be solved using the gradient method. We will consider this problem graphically. In the figure on the last page a number of level curves of f are depicted.

Assume that we start in the point $x^{(0)} = (1.41, 0.54)$ that has been marked with a ring in the graph. Draw in the graph how the gradient method would search and step in the three first three iterations if we assume that exact line search is used. If needed you may add extra level curves in the graph. Describe also in words how you see which direction that the gradient method determines and how you can tell how far you should go in that direction. Do the same thing for Newton's method. Attach the last page with the graph to your solutions. (3p)

Remark: Note that the scale on both axis is the same.

(b) Show using LDL^T-factorization that the problem is actually convex. . . . . (2p)

(c) Determine the exact solution to the quadratic (QP) problem. . . . . (2p)

(d) Assume that we add the linear constraint $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \end{bmatrix}.$$

Illustrate this constraint in the graph and determine with the help of the nullspace method or the Lagrange method the optimal solution to this constrained optimization problem. . . . . (3p)

5. Consider the nonlinear optimization problem

$$(P) \quad \left[\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{s.t.} & x \in \mathcal{F} \end{array} \right]$$

where the objective function is given by $f(\mathbf{x}) = e^{x_1+x_2} - x_1 - 3x_2$, and the feasible region is $\mathcal{F} = \mathcal{C}_1 \cap \mathcal{C}_2$ where

$$\mathcal{C}_1 = \left\{ \mathbf{x} \in \mathbb{R}^2 \; : \; e^{x_1} - x_2 \geq 0 \right\}$$

$$\mathcal{C}_2 = \left\{ \mathbf{x} \in \mathbb{R}^2 \; : \; e \cdot x_1^2 - x_2 \leq 0 \right\}.$$

(a) Is the objective function convex ?
Is the feasible region convex?
Is this a convex optimization problem?
Motivate your answers well. . . . . (3p)

(b) If we in a general situation have that a set $\mathcal{F}$ that is given by the intersection of two convex sets $\mathcal{C}_1$ and $\mathcal{C}_2$, what do we know about the convexity of $\mathcal{F}$?
If the two sets $\mathcal{C}_1$ and $\mathcal{C}_2$ are not convex, what do we then know about the convexity of $\mathcal{F}$? . . . . (2p)

(c) Does the point $\mathbf{x}^{(c)} = (0, 0)$ satisfy the KKT-conditions ? Can we say that this point is a local optimum based on what we determined so far? . . . . (2p)

(d) Does the point $\mathbf{x}^{(d)} = (1, e)$ satisfy the KKT-conditions ? Can we say that this point is a local optimum based on what we determined so far? . . . . (2p)

(e) Are there any interior point that satisfy the KKT-conditions? . . . . (2p)

Good Luck!

Take out and attach this page to your solutions.

Name:

Page number: