KTH Mathematics

**Solutions for the exam in Optimization.**
**tuesday August 25, 2009, time. 14.00–19.00**

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*There may be alternative solutions to the problem.*

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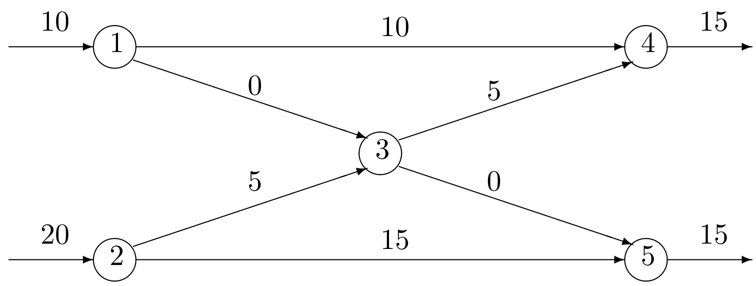
**1.** (a) The network can be described by the graph




where the optimal distribution plan is depicted. It is clearly feasible and corresponds to a spanning tree and basic solution.

The costs in the network could represent the cost of gasoline for driving the distance between physical truck stations, but also include toll expenses, driver salary, environmental fees and so on.

Put the node potential $y_5$ at node 5 to be 0. Then $y_2 - y_5 = c_{25}$ gives $y_2 = 2$.
Then $y_2 - y_3 = c_{23}$ gives $y_3 = 0$.
Then $y_3 - y_4 = c_{34}$ gives $y_4 = -2$.
Then $y_1 - y_4 = c_{14}$ gives $y_1 = 0$.

The reduced costs are now $r_{13} = c_{13} - y_1 + y_3 = 2$ and $r_{35} = c_{35} - y_3 + y_5 = 2$. (i.e. the total cost will increase with 2 units per lorry that takes the route through arc 13 or 35) Since the reduced costs are positive this verifies that the solution above is optimal.

(b) The problem $(P)$ can be written on standard form

$$(P_s) \quad \begin{bmatrix} \min\limits_x & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 \\ 2 & -2 & 3 & -1 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & -1 & -1 & 3 & 0 & 0 \end{bmatrix}^T.$$

(c) Assume that $x$ and $y$ are feasible for the primal and dual respectively. We have then that $b^T y = (Ax)^T y = x^T (A^T y) \leq x^T c = c^T x$ and the inequality followed since $x \geq 0$ and $A^T y \leq c$.

**2.** (a) The standard form is

$$(P_s) \quad \left[\begin{array}{ll} \min\limits_{x} & x_1 - 2x_2 \\ \text{s.t.} & x_1 + 3x_2 + x_3 = 1 \\ & x_1 - x_2 + x_4 = 1 \\ & x \geq 0 \end{array}\right]$$

We start with $x_1$ and $x_2$ in the basis, that is basic and non-basic variable indices are given by $\beta = \{1, 2\}$ and $\eta = \{3, 4\}$, so

$$B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\bar{b} = B^{-1}b = [4/7\ 1/7]^T$ giving the starting basic solution $x = (4/7, 1/7, 0, 0)$. Then the equations $B^T y = c_B$ and $\hat{c}_N^T = c_N^T - y^T N$ gives

$$y = \begin{bmatrix} -3/7 \\ 5/7 \end{bmatrix}, \quad \hat{c}_N^T = \begin{bmatrix} 3/7 & -5/7 \end{bmatrix}.$$

Let $x_4$ enter the basis. Which one should exit ?
From $B\hat{a}_4 = a_4$, we get $\hat{a}_4 = 1/7 * (3, -1)^T$, and since the second element is negative, $x_1$ exits the basis.
Update the basis and nonbasis matrices: Basic and non-basic variable indices are given by $\beta = \{2, 4\}$ and $\eta = \{1, 3\}$, and

$$B = \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Then the equations $B^T y = c_B$ and $\hat{c}_N^T = c_N^T - y^T N$ gives

$$y = \begin{bmatrix} -2/3 \\ 0 \end{bmatrix}, \quad \hat{c}_N^T = \begin{bmatrix} 5/3 & 2/3 \end{bmatrix}.$$

Since all reduced costs are nonnegative, the current bfs $\hat{x} = (0, 1/3, 0, 4/3)^T$ is optimal.

(b) First determine the nullspace matrix of $A$.
Perform row-operations to obtain

$$\begin{bmatrix} 1 & 0 & 1/7 & 3/7 \\ 0 & 1 & 2/7 & -1/7 \end{bmatrix},$$

then

$$Z = \begin{bmatrix} -1/7 & -3/7 \\ -2/7 & 1/7 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\hat{x} - \bar{x} = \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 4/3 \end{bmatrix} - \begin{bmatrix} 4/7 \\ 1/7 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4/7 \\ 4/21 \\ 0 \\ 4/3 \end{bmatrix} = \begin{bmatrix} -1/7 & -3/7 \\ -2/7 & 1/7 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4/3 \end{bmatrix} = Zv.$$

(The $v$ you get depends on which $Z$ you use)

(c) You need to check if there is a solution to the equation $A^T x = c$, i.e.

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

But this equation system clearly has no solution.

(d) We know that $\mathcal{R}(A^T)^\perp = \mathcal{N}(A)$, and since $Ac = [-5\ 4]^T$ the vector $c$ is not in the nullspace of $A$.

**3.** (a) For $f$ to be convex on the whole $\mathbb{R}^3$ it is necessary that the matrix $H$ is positive semidefinite. Use $LDL^T$-factorization:

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since all the diagonal elements in the $D$-matrix are positive, the matrix $H$ is positive definite.

(b) With the nullspace method a $Z$-matrix and $\bar{x}$ given by

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

the equation system $(ZHZ^T)v = -Z^T(H\bar{x} + c)$:

$$\begin{bmatrix} 1 & -1 \\ -1 & 15 \end{bmatrix} v = \begin{bmatrix} 1 \\ -15 \end{bmatrix},$$

yields $v = [0\ \ -1]^T$. Therefore, $\hat{x} = \bar{x} + Zv = [0\ 1\ 0]^T$ is a global minimum since the problem is convex.

(c) For the Lagrange method, the following equation system must be solved

$$\begin{bmatrix} H & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix},$$

That is:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & -1 \\ 1 & 0 & 10 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \hat{u} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \\ 1 \end{bmatrix},$$

from which we see that $\hat{u} = 4$.

**4.** (a) For the optimization problem to be convex, it is necessary that the feasible region is convex and that the objective function is convex on the whole feasible region.

The objective function is convex since it is the sum of the convex functions

$$f_1(x) = e^{x_1+x_3} - x_1 - x_3, \quad f_2(x) = x_2^4 + 4x_2.$$

whose convexity follows from

$$\nabla^2 f_1(x) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \geq 0 \forall x, \quad \nabla^2 f_1(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12x_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0 \forall x.$$

The feasible region is not convex since it is given by non-linear equality constraints where, for example, the points $x^{(1)} = (1,0,0)$ and $x^{(2)} = (-1,0,0)$ are feasible, but $x^{(a)} = 1/2x^{(1)} + 1/2x^{(2)}$ is not feasible.

Therefore, the optimization problem is not convex.

(b) The points $x^{(1)} = (1,0,0)$, $x^{(2)} = (-1,0,0)$ and $x^{(3)} = (0,-1,0)$ are feasible points with the third coordinate equal to zero.

(c) Let $h_1(x) = x_1^2 + x_2^2 + x_3^2 - 1$ and $h_2(x) = x_1^2 - 1 - x_2$, then

$$\nabla f(x) = \begin{bmatrix} e^{x_1+x_3} - 1 & 4x_2^3 + 4 & e^{x_1+x_3} - 1 \end{bmatrix},$$

$$\nabla h_1(x) = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \end{bmatrix}, \quad \nabla h_2(x) = \begin{bmatrix} 2x_1 & -1 & 0 \end{bmatrix}.$$

In $x^{(1)}$, we have $\nabla f(x^{(1)}) + u_1 \nabla h_1(x^{(1)}) + u_2 \nabla h_2(x^{(1)})$ equal to

$$\begin{bmatrix} e - 1 + 2u_1 + 2u_2 & 4 + 0u_1 + (-1)u_2 & e - 1 + 0u_1 + 0u_2 \end{bmatrix},$$

which can never be zero.

In $x^{(2)}$, we have $\nabla f(x^{(2)}) + u_1 \nabla h_1(x^{(2)}) + u_2 \nabla h_2(x^{(2)})$ equal to

$$\begin{bmatrix} 1/e - 1 - 2u_1 - 2u_2 & 4 + 0u_1 - 1u_2 & 1/e - 1 + 0u_1 + 0u_2 \end{bmatrix},$$

which can never be zero.

In $x^{(3)}$, we have $\nabla f(x^{(3)}) + u_1 \nabla h_1(x^{(3)}) + u_2 \nabla h_2(x^{(3)})$ equal to

$$\begin{bmatrix} 1 - 1 + 0u_1 + 0u_2 & -4 + 4 + (-2)u_1 - 1u_2 & 1 - 1 + 0u_1 + 0u_2 \end{bmatrix},$$

which is zero for $u_1 = u_2 = 0$.

So only the last point satisfies the KKT-conditions.

(d) The only point that could be globally optimal is $x^{(3)}$, since it is the only one satisfying the KKT-conditions. But the problem is not convex, so the KKT-conditions are not sufficent. However, the gradient of $f$ is zero at $x^{(3)}$ and the objective function is convex on the whole $\mathbb{R}^3$ so therefore it is optimal on the whole space and hence also optimal on the constrained space.

**5.** (a) For the optimization problem to be convex, it is necessary that the feasible region is convex and that the objective function is convex on the whole feasible region.

The objective function is not convex since the part depending on $x_2$ is in fact strictly concave. With the functions

$$g_1(x) = x_1^2 + x_2^2 - 1 \quad g_2(x) = x_1^2 - 1 - x_2$$

the constraints can be written $g_1(x) \leq 0$ and $g_2(x) \leq 0$, and since these functions are convex the constraints define a convex feasible region.

However, the optimization problem is not convex.

(b) At the point $x^{(b)}$ both the constraints are active ($g_i(x^{(b)}) = 0$ for $i = 1, 2$), which shows that the point is feasible and that both Lagrange parameters may be non-zero.

The gradient of $f$ is given by

$$\nabla f(x) = \begin{bmatrix} -e^{-x_1} & -2x_2 \end{bmatrix}.$$

At the given point

$$\nabla f(x^{(b)})^T = \begin{pmatrix} -1/e \\ 0 \end{pmatrix},$$

$$\nabla g_1(x^{(b)})^T = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \nabla g_2(x^{(b)})^T = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

For the KKT-conditions to be satisfied we need to find non-negative Lagrange parameters such that

$$\begin{pmatrix} -1/e \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} y_1 + \begin{pmatrix} 2 \\ -1 \end{pmatrix} y_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now $y_1 = 1/(2e)$ and $y_2 = 0$. So the KKT-conditions are satisfied.

But since the problem is not convex, we can not say that $x^{(b)}$ is a local minimum based on this. (In fact it is not a local minimum)

(c) Along the arc of the circle in the positive orthant the constraint $g_1$ is active and the other is not. (except for the point considered in (b))

That is, the equation $x_1^2 + x_2^2 = 1$ must hold and also the following KKT conditions, $y_2 = 0$ and

$$\begin{pmatrix} -e^{-x_1} \\ -2x_2 \end{pmatrix} + \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} y_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, if $x_2 \neq 0$ then $y_1 = 1$ and $-e^{-x_1} + 2x_1 = 0$, giving us the equation that $x_1$ must solve, namely $e^{-x_1} = 2x_1$ which clearly as a solution in the interval $(0, 1)$. (if $x_2 = 0$ then we get the point $x^{(b)}$)

(d) This can be done in a number of ways. First note that the region is a compact convex set and the objective function is continuous, so we know that at least one optimal point exists. One approach is to find all KKT-points and compare the values at those points. Symmetry of the feasible region can be used to deduce that there must be an optimal point in the positive orthant. In fact, the feasible region is symmetric in the $x_1$ variable, but for each negative $x_1$ the objective function take a larger value than for the corresponding positive value $-x_1$. Similarly, the objective function takes the same value for a positive

$x_2$ and the corresponding negative value $-x_2$, but if the negative value $-x_2$ is feasible, then $x_2$ is also feasible and therefore an optimal positive $x_2$ can be found. The gradient of $f$, $\nabla f(x) = (-e^{-x_1},\ -2x_2)$, is non-zero for all $x$ in the positive orthant, so no interior point of the feasible region can be optimal, hence the optimum is obtained at the boundary. Along the arc we have that $f(x) = e^{-x_1} - 1 + x_1^2$, which is a convex function and assumes it minimum when the derivative is zero, i.e. at the point in (c).