Solutions to exam in SF1811 Optimization, August 18, 2014

1.(a)

The considered LP problem is a minimum cost network flow problem with three nodes: 1, 2 and 3, and six arcs: (1,2), (2,1), (1,3), (3,1), (2,3) and (3,2).

The suggested solution $\hat{\mathbf{x}} = (15, 0, 10, 0, 0, 0)^\mathsf{T}$ corresponds to a spanning tree with the arcs (1,2) and (1,3), i.e. a basic solution. It is a feasible basic solution since all the balance equations (in all nodes) are satisfied and all variables are non-negative.

The simplex variables y_i are obtained from the equations $y_i - y_j = c_{ij}$ for basic variables, together with $y_3 = 0$. This gives
$y_3 = 0$,
$y_1 = y_3 + c_{13} = 0 + 2 = 2$,
$y_2 = y_1 - c_{12} = 2 - 3 = -1$.

Then the reduced costs for the non-basic variables are obtained from $r_{ij} = c_{ij} - y_i + y_j$:
$r_{21} = 1 - (-1) + 2 = 4$,
$r_{31} = 1 - 0 + 2 = 3$,
$r_{23} = 1 - (-1) + 0 = 2$,
$r_{32} = 1 - 0 + (-1) = 0$.

Since all $r_{ij} \geq 0$, the suggested solution is optimal.

However, since $r_{32} = 0$, the objective value will not change if we let x_{32} become a new basic variable. Let $x_{32} = t$ and increase t from 0. Then the basic variables will change according to $x_{12} = 15 - t$ and $x_{13} = 10 + t$.
In particular, with $t = 15$, we obtain a new optimal basic solution $\tilde{\mathbf{x}} = (0, 0, 25, 0, 0, 15)^\mathsf{T}$.

Check: $\mathbf{c}^\mathsf{T}\hat{\mathbf{x}} = 3 \cdot 15 + 2 \cdot 10 = 65$. $\mathbf{c}^\mathsf{T}\tilde{\mathbf{x}} = 2 \cdot 25 + 1 \cdot 15 = 65$.

1.(b)

We apply Gauss–Jordan's method on the given matrix $\mathbf{B} = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 16 & 32 \\ 64 & 128 & 256 \end{bmatrix}$.

Add -8 times the first row to the second row and -64 times the first row to the third row.

Then the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is obtained, and $\mathbf{B}$ has been transformed to *reduced row echelon form* with only one *leading one*: $\mathbf{U} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$.

Note that $\mathcal{N}(\mathbf{B}^\mathsf{T})^\perp = \mathcal{R}(\mathbf{B})$, and that a basis to $\mathcal{R}(\mathbf{B})$ is obtained by chosing the columns in $\mathbf{B}$ corresponding to the "leading ones" in $\mathbf{U}$, i.e. the first column in $\mathbf{B}$.

Thus, the single vector $\begin{pmatrix} 1 \\ 8 \\ 64 \end{pmatrix}$ forms a basis to $\mathcal{R}(\mathbf{B})$, and thus also to $\mathcal{N}(\mathbf{B}^\mathsf{T})^\perp$.

In order to find a basis for $\mathcal{N}(\mathbf{B})$, note that the system $\mathbf{Bx} = \mathbf{0}$ is equivalent to the system $\mathbf{Ux} = \mathbf{0}$, i.e. $x_1 + 2x_2 + 4x_3 = 0$, for which the general solution is obtained by letting $x_2 = t$ and $x_3 = s$, where t and s are arbitrary real numbers. Then $x_1 = -2t - 4s$, and the general solution to $\mathbf{Bx} = \mathbf{0}$ can thus be written $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$.

It follows that the two vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $\mathcal{N}(\mathbf{B})$.

By repeating the above steps on $\mathbf{B}^\mathsf{T}$ instead of $\mathbf{B}$ the following is obtained:

The single vector $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ forms a basis to $\mathcal{R}(\mathbf{B}^\mathsf{T})$, and thus also to $\mathcal{N}(\mathbf{B})^\perp$.

The two vectors $\begin{pmatrix} -8 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -64 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $\mathcal{N}(\mathbf{B}^\mathsf{T})$.

Check of orthogonality:

$$\begin{pmatrix} 1 \\ 8 \\ 64 \end{pmatrix}^\mathsf{T} \begin{pmatrix} -8 \\ 1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ 8 \\ 64 \end{pmatrix}^\mathsf{T} \begin{pmatrix} -64 \\ 0 \\ 1 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}^\mathsf{T} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}^\mathsf{T} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} = 0.$$

The vector $(1, b, 1)^\mathsf{T}$ belong to $\mathcal{N}(\mathbf{B})$ if and only if $[\,1\ 2\ 3\,](1, b, 1)^\mathsf{T} = 0$,
i.e. if and only if $1 + 2b + 4 = 0$, i.e. if and only $b = -5/2$.

2.(a) Introduce the following new non-negative variables x'_j:
$x'_1 = x_1,\ \ x'_2 = x_2,\ \ x'_3 - x'_4 = x_3,$
$x'_5 =$ slack variable for the constraint $x_1 - x_2 + x_3 \geq 0,$
$x'_6 =$ slack variable for the constraint $x_2 + x_3 \geq 0.$
Further, introduce the variable vector $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4, x'_5, x'_6)^\mathsf{T}$.

Then the problem can be written as the following LP problem on standard form:

$$\begin{aligned} \text{minimize}\quad & \mathbf{c}^\mathsf{T}\mathbf{x}' \\ \text{subject to}\quad & \mathbf{A}\mathbf{x}' = \mathbf{b}, \quad \mathbf{x}' \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{c} = (0, 0, 1, -1, 0, 0)^\mathsf{T}$.

2.(b) and **2.(c)**
The suggested solution $\hat{\mathbf{x}} = (2,\ 1, -1)^\mathsf{T}$ corresponds to the solution $\hat{\mathbf{x}}' = (2,\ 1, 0, 1, 0, 0)^\mathsf{T}$ to the above problem on standard form. The optimality of this solution can be verified by showing that $\hat{\mathbf{x}}'$ is a feasible basic solution with non-negative reduced costs.
Alternatively, the optimality of $\hat{\mathbf{x}} = (2,\ 1, -1)^\mathsf{T}$ can be verified using the complementarity theorem. This is the approach used here, and then 2.(c) is simultaneously solved.
When the primal problem is

$$\begin{aligned} \text{P:}\ \text{minimize}\quad & x_3 \\ \text{subject to}\quad & x_1 - x_2 + x_3 \geq 0, \\ & x_2 + x_3 \geq 0, \\ & x_1 + x_2 = 3, \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \text{ "free"}, \end{aligned}$$

the corresponding dual problem is

$$\begin{aligned} \text{D:}\ \text{maximize}\quad & 3y_3 \\ \text{subject to}\quad & y_1 + y_3 \leq 0, \\ & -y_1 + y_2 + y_3 \leq 0, \\ & y_1 + y_2 = 1, \\ & y_1 \geq 0,\ y_2 \geq 0,\ y_3 \text{ "free"}. \end{aligned}$$

The complementary theorem says that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are optimal solutions to P and D, respectively, if and only if

(1) $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are feasible solutions to P and D,

(2) $\hat{y}_1(\hat{x}_1 - \hat{x}_2 + \hat{x}_3) = 0,\ \hat{y}_2(\hat{x}_2 + \hat{x}_3) = 0,\ \hat{x}_1(\hat{y}_1 + \hat{y}_3) = 0$ and $\hat{x}_2(-\hat{y}_1 + \hat{y}_2 + \hat{y}_3) = 0$.

Since the suggested point $\hat{\mathbf{x}} = (2,\ 1, -1)^\mathsf{T}$ is a feasible solution to P, it is an optimal solution to P if and only if there is a feasible solution $\hat{\mathbf{y}}$ to D such that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ satisfy (2) above.

Note that $\hat{\mathbf{x}}$ satisfies $\hat{x}_1 - \hat{x}_2 + \hat{x}_3 = 0,\ \hat{x}_2 + \hat{x}_3 = 0,\ \hat{x}_1 + \hat{x}_2 = 3,\ \hat{x}_1 > 0$ and $\hat{x}_2 > 0$.
Thus, $\hat{\mathbf{y}}$ must satisfy $\hat{y}_1 + \hat{y}_3 = 0,\ -\hat{y}_1 + \hat{y}_2 + \hat{y}_3 = 0,\ \hat{y}_1 + \hat{y}_2 = 1,\ \hat{y}_1 \geq 0$ and $\hat{y}_2 \geq 0$.
The unique solution to the first three equations is $\hat{\mathbf{y}} = (1/3,\ 2/3,\ -1/3)^\mathsf{T}$, and since this solution satisfies $\hat{y}_1 \geq 0$ and $\hat{y}_2 \geq 0$, it follows that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ satisfy (1) and (2) above.

Thus, $\hat{\mathbf{x}} = (2,\ 1, -1)^\mathsf{T}$ and $\hat{\mathbf{y}} = (1/3,\ 2/3,\ -1/3)^\mathsf{T}$ are optimal solutions to P and D.
The optimal value of P $= \hat{x}_3 = -1$. The optimal value of D $= 3\hat{y}_3 = 3 \cdot (-1/3) = -1$.

3.(a)

The objective function is $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{x}$, with $\mathbf{H} = \begin{bmatrix} 2 & -3 & -3 \\ -3 & 2 & -3 \\ -3 & -3 & 2 \end{bmatrix}$, $\mathbf{c} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}$.

LDL^T-factorization of $\mathbf{H}$ gives

$$\mathbf{H} = \mathbf{LDL}^\mathsf{T} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -1.5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2.5 & 0 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} 1 & -1.5 & -1.5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Since there is a negative diagonal element in $\mathbf{D}$, the matrix $\mathbf{H}$ is *not* positive semidefinite, which in turn implies that there is no optimal solution to the problemen of minimizing $f(\mathbf{x})$ without constraints. (With e.g. $\mathbf{d} = (1,1,1)^\mathsf{T}$, $f(t\,\mathbf{d}) = -12t^2 + 60t \to -\infty$ when $t \to \infty$.)

3.(b)

We now have a QP problem with equality constraints, i.e. a problem of the form

minimize $\frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$,

where $\mathbf{A} = [\,1 \;\; 1 \;\; 1\,]$, $\mathbf{b} = 3$, $\mathbf{H} = \begin{bmatrix} 2 & -3 & -3 \\ -3 & 2 & -3 \\ -3 & -3 & 2 \end{bmatrix}$ and $\mathbf{c} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}$.

The general solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$, i.e. to $x_1 + x_2 + x_3 = 3$, is given by

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot v_1 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot v_2$, for arbitrary values on v_1 and v_2,

which means that $\bar{\mathbf{x}} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ is a feasible solution, and $\mathbf{Z} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a matrix whos columns form a basis for the null space of $\mathbf{A}$.

After the variable change $\mathbf{x} = \bar{\mathbf{x}} + \mathbf{Z}\mathbf{v}$ we should solve the system $(\mathbf{Z}^\mathsf{T}\mathbf{H}\mathbf{Z})\mathbf{v} = -\mathbf{Z}^\mathsf{T}(\mathbf{H}\bar{\mathbf{x}} + \mathbf{c})$, provided that $\mathbf{Z}^\mathsf{T}\mathbf{H}\mathbf{Z}$ is at least positive semidefinite.

We have $\mathbf{Z}^\mathsf{T}\mathbf{H}\mathbf{Z} = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}$, which is positive definite (since $10 > 0$, $10 > 0$, $10 \cdot 10 - 5 \cdot 5 > 0$).

The system $(\mathbf{Z}^\mathsf{T}\mathbf{H}\mathbf{Z})\mathbf{v} = -\mathbf{Z}^\mathsf{T}(\mathbf{H}\bar{\mathbf{x}} + \mathbf{c})$ becomes

$\begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$, with the unique solution $\hat{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which implies that

$\hat{\mathbf{x}} = \bar{\mathbf{x}} + \mathbf{Z}\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ is the unique optimal solution to our problem.

4.(a)

The objective function is $f(\mathbf{x}) = (x_1^2 + x_2^2 + 1)^{1/2} - 0.3\,x_1 - 0.4\,x_2$.

The gradient of f becomes $\nabla f(\mathbf{x}) = \left(\dfrac{x_1}{(x_1^2 + x_2^2 + 1)^{1/2}} - 0.3,\ \dfrac{x_2}{(x_1^2 + x_2^2 + 1)^{1/2}} - 0.4 \right)$.

The Hessian of f becomes $\mathbf{F}(\mathbf{x}) = \dfrac{1}{(x_1^2 + x_2^2 + 1)^{3/2}} \cdot \begin{bmatrix} 1 + x_2^2 & -x_1 x_2 \\ -x_1 x_2 & 1 + x_1^2 \end{bmatrix}$.

The starting point is given by $\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and then

$f(\mathbf{x}^{(1)}) = 1$, $\nabla f(\mathbf{x}^{(1)}) = (-0.3, -0.4)$ and $\mathbf{F}(\mathbf{x}^{(1)}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since a diagonal matrix with strictly positive diagonal elements is positive definite, the Hessian $\mathbf{F}(\mathbf{x}^{(1)})$ is positive definite, and then the first Newton search direction $\mathbf{d}^{(1)}$ is obtained by solving the system

$\mathbf{F}(\mathbf{x}^{(1)})\mathbf{d} = -\nabla f(\mathbf{x}^{(1)})^{\mathsf{T}}$, i.e. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{d} = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$, with the solution $\mathbf{d}^{(1)} = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$.

First try $t_1 = 1$, so that $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + t_1 \mathbf{d}^{(1)} = \mathbf{x}^{(1)} + \mathbf{d}^{(1)} = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$.

Then $f(\mathbf{x}^{(2)}) = \sqrt{1.25} - 0.09 - 0.16 < 1.2 - 0.25 < 1 = f(\mathbf{x}^{(1)})$,
so $t_1 = 1$ is accepted, and the first iteration is completed.

4.(b)

The function f is convex on $I\!R^2$ if and only if the Hessian

$$\mathbf{F}(\mathbf{x}) = \frac{1}{(x_1^2 + x_2^2 + 1)^{3/2}} \cdot \begin{bmatrix} 1 + x_2^2 & -x_1 x_2 \\ -x_1 x_2 & 1 + x_1^2 \end{bmatrix} \text{ is positive semidefinite for all } \mathbf{x} \in I\!R^2,$$

which holds if and only if $\begin{bmatrix} 1 + x_2^2 & -x_1 x_2 \\ -x_1 x_2 & 1 + x_1^2 \end{bmatrix}$ is positive semidefinite for all $\mathbf{x} \in I\!R^2$.

But $1 + x_2^2 > 0$, $1 + x_1^2 > 0$, and $(1 + x_2^2)(1 + x_1^2) - (-x_1 x_2)(-x_1 x_2) = 1 + x_1^2 + x_2^2 > 0$ for all $\mathbf{x} \in I\!R^2$, which implies that $\mathbf{F}(\mathbf{x})$ is in fact positive definite for all $\mathbf{x} \in I\!R^2$, which in turn implies that f is strictly convex on the whole set $I\!R^2$.

4.(c)

We should solve $\nabla f(\mathbf{x}) = (0, 0)$, i.e. $\dfrac{x_1}{(x_1^2 + x_2^2 + 1)^{1/2}} = 0.3$ and $\dfrac{x_2}{(x_1^2 + x_2^2 + 1)^{1/2}} = 0.4$.

Some analytical calculations show that the only solution to this system is

$$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)^{\mathsf{T}} = \left(\frac{0.6}{\sqrt{3}}, \frac{0.8}{\sqrt{3}} \right)^{\mathsf{T}}.$$

Since f is strictly convex on $I\!R^2$, $\hat{\mathbf{x}}$ is the unique globally optimal solution to the problem of minimizing $f(\mathbf{x})$ on $I\!R^2$.

5. With $f(\mathbf{x}) = \sum_{j=1}^{n} \dfrac{c_j}{1-x_j}$ and $g(\mathbf{x}) = \sum_{j=1}^{n} \dfrac{1}{1+x_j} - n$, the Lagrange function becomes

$$L(\mathbf{x}, y) = f(\mathbf{x}) + y\,g(\mathbf{x}) = \sum_{j=1}^{n} \frac{c_j}{1-x_j} + y\,(\sum_{j=1}^{n} \frac{1}{1+x_j} - n) = -yn + \sum_{j=1}^{n} (\,\frac{c_j}{1-x_j} + \frac{y}{1+x_j}\,).$$

The Lagrange relaxed problem PR_y is defined, for a given $y \geq 0$,
as the problem of minimizing $L(\mathbf{x}, y)$ with respect to $\mathbf{x} \in X$.
But this problem separates into one problem for each variable x_j, namely

$$\text{minimize } \ell_j(x_j) = \frac{c_j}{1-x_j} + \frac{y}{1+x_j} \text{ subject to } -1 < x_j < 1. \tag{0.1}$$

We have that $\ell_j'(x_j) = \dfrac{c_j}{(1-x_j)^2} - \dfrac{y}{(1+x_j)^2}$ and $\ell_j''(x_j) = \dfrac{2c_j}{(1-x_j)^3} + \dfrac{2y}{(1+x_j)^3} > 0$,
which implies that $\ell_j(x_j)$ is strictly convex on the interval $(-1,\, 1\,)$.

In accordance to the instructions, we will from now on only consider the case $y > 0$.

Then there is a unique solution $\tilde{x}_j(y)$ to the equation $\ell_j'(x_j) = 0$, namely

$$\tilde{x}_j(y) = \frac{\sqrt{y} \, - \sqrt{c_j}}{\sqrt{y} \, + \sqrt{c_j}}, \tag{0.2}$$

which belongs to the interval $(-1,\, 1\,)$ for all $y > 0$.
We conclude that this $\tilde{x}_j(y)$ is the unique optimal solution to the subproblem (**??**).

The dual objective function is then given by

$$\varphi(y) = L(\tilde{\mathbf{x}}(y), y) = -yn + \sum_{j=1}^{n} (\,\frac{c_j}{1-\tilde{x}_j(y)} + \frac{y}{1+\tilde{x}_j(y)}\,) = -yn + \tfrac{1}{2} \sum_{j=1}^{n} (\sqrt{y} + \sqrt{c_j}\,)^2.$$

Then $\varphi'(y) = -n + \dfrac{1}{2\sqrt{y}} \sum_{j=1}^{n} (\sqrt{y} + \sqrt{c_j}\,) = -\dfrac{n}{2} + \dfrac{1}{2\sqrt{y}} \sum_{j=1}^{n} \sqrt{c_j}$

and $\varphi''(y) = -\dfrac{1}{4y\sqrt{y}} \sum_{j=1}^{n} \sqrt{c_j} < 0$ for all $y > 0$, so that φ is strictly concave when $y > 0$.

Assume from now on that $n = 3$, $c_1 = 1$, $c_2 = 4$ and $c_1 = 9$.

Then $\varphi'(y) = -\dfrac{3}{2} + \dfrac{6}{2\sqrt{y}}$ and the unique solution to $\varphi'(y) = 0$ is $\hat{y} = 4$.
Since φ is strictly concave for $y > 0$ it follows that $\varphi(4) > \varphi(y)$ for all $y > 0$.

The corresponding primal solution is $\hat{\mathbf{x}} = (\tilde{x}_1(4),\, \tilde{x}_2(4),\, \tilde{x}_3(4))^{\mathsf{T}} = (1/3,\, 0,\, -1/5)^{\mathsf{T}}$,
which satisfies $g(\hat{\mathbf{x}}) = \dfrac{1}{1+1/3} + \dfrac{1}{1+0} + \dfrac{1}{1-1/5} - 3 = 0$.

It follows that $\hat{\mathbf{x}} = (1/3,\, 0,\, -1/5)^{\mathsf{T}}$ and $\hat{y} = 4$ satisfy the global optimality conditions,
and thus $\hat{\mathbf{x}}$ is a global optimal solution to the primal problem.

Since X is a convex set and $g(\mathbf{x})$ is a convex function on X, the feasible region for the primal problem is a convex set. Since, in addition, $f(\mathbf{x})$ is a *strictly* convex function on X, it follows that the obtained optimal solution $\hat{\mathbf{x}}$ must be the *unique* optimal solution to the primal problem P.