Exercise 22.2

Denote by $S(X)$ the intersection of all subspaces that contain X. (The set of subspaces containing X is not empty, since for example $\mathbb{R}^n$ contains X.)

S1 $0 \in S(X)$, since every subspace contains it.

S2 Suppose $v_1, v_2 \in S(X)$. Then if S' is a subspace containing X, $v_1, v_2 \in S(X) \subset S'$. Since S' is a subspace, $v_1 + v_2 \in S'$. So $v_1 + v_2$ belongs to every subspace containing X. Hence $v_1 + v_2 \in S(X)$

S3 Let $\alpha \in \mathbb{R}$ and $v \in S(X)$. If S' is a subspace containing X, then $v \in S(X) \subset S'$. Since S' is a subspace, $\alpha . x \in S'$. So $\alpha . x$ belongs to every subspace containing X. Hence $\alpha \cdot x \in S(X)$.

So $S(X)$ is a subspace.

Clearly $S(X)$ contains X since it is the intersection of all subspaces containing X.

Suppose S' is a subspace containing X. Then $S(X) \subset S'$. So $S(X)$ is the smallest subspace of $\mathbb{R}^n$ that contains X.

$\text{span}\, \emptyset = \{0\}$: Since $\emptyset \subset \{0\}$, we have $\text{span}\, \emptyset \subset \{0\}$. But also $0 \in \text{span}\, \emptyset$, and so $\text{span}\, \emptyset = \{0\}$.

$X \subset \text{span}\, X$, and so $v_1, \dots, v_k \in \text{span}\, X$. Since $\text{span}\, X$ is a subspace, also $\alpha_1 \cdot v_1 + \dots + \alpha_k \cdot v_k \in \text{span}\, X$ for all scalars $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. So $\{\alpha_1 v_1 + \dots + \alpha_k \cdot v_k : \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$ is contained in $\text{span}\, X$. It is clear that $\{\alpha_1 . v_1 + \dots + \alpha_k . v_k : \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$ containin $v_1, \dots, v_k$. So $\text{span}\, X \subset \{\alpha_1 v_1 + \dots + \alpha_k . v_k : \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$

Exercise 22.3

S1. $0^T = 0$, and so the zero matrix $0 \in \mathbb{R}^{n \times n}$ belongs to S.

S2. Suppose $A, B \in S$. Then $A^T = A$, $B^T = B$.
Then $(A+B)^T = A^T + B^T = A + B$, and so $A + B \in S$.

S3. If $A \in S$ and $\alpha \in \mathbb{R}$, then $(\alpha A)^T = \alpha A^T = \alpha A$ and so $\alpha A \in S$.

So S is a subspace of $\mathbb{R}^{n \times n}$.

Exercise 22.4

span $\emptyset = \{0\}$ (Exercise 23.2), and $\emptyset$ is linearly independent.
Thus it forms a basis for $\{0\}$

Exercise 22.5

Let E_{ij} be the $n \times n$ matrix with 1 in the ith row and jth column, and all other entries equal to 0.
Let $B = \{ E_{ij} + E_{ji} : 1 \leq i < j \leq n \} \cup \{ E_{ii} : 1 \leq i \leq n \}$.
We claim that B forms a basis for S.
Suppose that $A \in S$. Since $A = A^T$, A has the form

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$= a_{11} E_{11} + \dots + a_{nn} E_{nn} + a_{12} (E_{12} + E_{21}) + \dots + a_{1n} (E_{1n} + E_{n1})$$
$$+ a_{23} (E_{23} + E_{32}) + \dots + a_{2n} (E_{2n} + E_{n2})$$
$$+ a_{34} (E_{34} + E_{43}) + \dots + a_{3n} (E_{3n} + E_{n3})$$
$$+ \cdots$$
$$+ a_{(n-1),n} (E_{(n-1),n} + E_{n,(n-1)}).$$

$\in$ span B.

Conversely, span B is clearly contained in S.
B is linearly independent since if
$$\alpha_{11} E_{11} + \dots + \alpha_{nn} E_{nn} + \sum_{1 \leq i < j \leq n} \alpha_{ij} (E_{ij} + E_{ji}) = 0,$$
then
$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{12} & \ddots & & \vdots \\ \vdots & & & \\ \alpha_{1n} & \cdots & & \alpha_{nn} \end{bmatrix} = 0,$$
and so $\alpha_{11} = \dots = \alpha_{nn} = \alpha_{ij} = 0$ for all $i < j$.

Thus B forms a basis for S.