Exercise 23.2

(1) First we show that $\ker(AA^T) = \ker A^T$.

If $x \in \ker A^T$, then $A^T x = 0$, and so

$AA^T x = A(A^T x) = A(0) = 0$. So $x \in \ker(AA^T)$.

Hence $\ker A^T \subset \ker(AA^T)$.

If $x \in \ker(AA^T)$, then $AA^T x = 0$ and so with

$y := A^T x$, we have

$y^T y = (A^T x)^T A^T x = x^T A A^T x = x^T 0 = 0$.

Hence $y = 0$ i.e., $A^T x = 0$, i.e., $x \in \ker A^T$.

So $\ker(AA^T) \subset \ker A^T$.

Consequently $\ker A^T = \ker(AA^T)$.

Finally, $\operatorname{ran}(AA^T) = (\ker (AA^T)^T)^\perp$

$= (\ker(AA^T))^\perp$

$= (\ker A^T)^\perp$

$= \operatorname{ran} A$.

(2) Since H is symmetric, $H = H^T$.

Hence $\operatorname{ran} H = (\ker H^T)^\perp = (\ker H)^\perp$.

Exercise 23.3

(1) If the columns of A span $\mathbb{R}^m$, then they form a basis for $\mathbb{R}^m$.

So $\dim(\operatorname{ran} A) = m$.

Hence $\dim(\operatorname{ran} A^T) = m$ as well.

Since the m columns of A^T span $\operatorname{ran} A^T$, and $\dim(\operatorname{ran} A^T) = m$, it follows that the m columns form a basis for A^T and in particular, they are linearly independent.

Now suppose the columns of A^T are linearly independent. Since the columns of A^T also span $\operatorname{ran} A^T$, it follows that they form a basis for $\operatorname{ran} A^T$. Hence $\dim(\operatorname{ran} A^T) = m$. Then also $\dim(\operatorname{ran} A) = m$. So we have $\operatorname{ran} A \subset \mathbb{R}^m$ and $\dim(\operatorname{ran} A) = m = \dim(\mathbb{R}^m)$. Hence $\operatorname{ran} A = \mathbb{R}^m$. So the columns of A span $\mathbb{R}^m$.

(2) Just replace A by A^T in (1).