Exercise 3.3.

(1) We introduce 4 slack variables, $\alpha, \beta, \gamma, \delta$, and the constraints for x, y, z are now replaced by the following:

$$x + y + \alpha = 3$$
$$-x - y + \beta = -2$$
$$x + z + \gamma = 5$$
$$-x - z + \delta = -4$$
$$x, y, z, \alpha, \beta, \gamma, \delta \geq 0.$$

So the problem is equivalent to the following linear programming problem in the standard form:

$$\text{minimize} \quad c^T X$$
$$\text{subject to} \quad AX = b$$
$$X \geq 0,$$

where $c = [1 \ 2 \ 3 \ 0 \ 0 \ 0 \ 0]^T \in \mathbb{R}^7$

$$b = \begin{bmatrix} 3 \\ -2 \\ 5 \\ -4 \end{bmatrix} \in \mathbb{R}^4$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 7}$$

and the variable $X = [x \ y \ z \ \alpha \ \beta \ \gamma \ \delta]^T$ takes values in $\mathbb{R}^7$.

(2) We set $\xi := x - 1$, $\eta := y - 2$, $\zeta := z - 1$.
Then the objective function is

$$x + y + z = (\xi + 1) + (\eta + 2) + (\zeta + 1) = \xi + \eta + \zeta + 4,$$

the constraint $x + 2y + 3z = 10$
becomes $\xi + 1 + 2\eta + 4 + 3\zeta + 3 = 10$, i.e., $\xi + 2\eta + 3\zeta = 2$.
Finally $[x \geq 1, y \geq 2, z \geq 1] \Leftrightarrow [\xi \geq 0, \eta \geq 0, \zeta \geq 0]$
So the problem is equivalent to the following linear programming problem in the standard form:

$$\begin{cases} \text{minimize} & \xi + \eta + \zeta \\ \text{subject to} & \xi + 2\eta + 3\zeta = 2, \\ & \xi \geq 0, \eta \geq 0, \zeta \geq 0. \end{cases}$$

(3) (Let $r \in \mathbb{R}$. Define $r_+ := \frac{r+|r|}{2}$ and $r_- := \frac{|r|-r}{2}$.
Then $r_+, r_- \geq 0$, $r = r_+ - r_-$ and $|r| = r_+ + r_-$.
Also if $r = r_+' - r_-'$, $|r| = r_+' + r_-'$ for some $r_+', r_-' \geq 0$, then adding the equations we get $2r_+' = r + |r|$ and so $r_+' = \frac{r+|r|}{2}$ and similarly $|r| - r = (r_+' + r_-') - (r_+' - r_-') = 2r_-'$ so that $r_-' = \frac{|r|-r}{2}$.)

The given problem is equivalent to the following linear programming problem in standard form:

$$(LP): \begin{cases} \text{minimize} & x_+ + x_- + y_+ + y_- + z_+ + z_- \\ \text{subject to} & x_+ - x_- + 2y_+ - 2y_- = 1, \\ & x_+ - x_- + z_+ - z_- = 1, \\ & x_+ \geq 0, \quad y_+ \geq 0, \quad z_+ \geq 0, \\ & x_- \geq 0, \quad y_- \geq 0, \quad z_- \geq 0, \end{cases}$$

First of all, if $(\hat{x}, \hat{y}, \hat{z})$ is optimal for the original problem (henceforth referred to as (P)), then $(\hat{x}_+, \hat{x}_-, \hat{y}_+, \hat{y}_-, \hat{z}_+, \hat{z}_-)$ is optimal for (LP). Indeed if $(\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2)$ belongs to the feasible set of (LP), then it is clear that (x, y, z), given by $x = \xi_1 - \xi_2$, $y = \eta_1 - \eta_2$, $z = \zeta_1 - \zeta_2$, belongs to the feasible set of (P). We have

$$\xi_1 + \xi_2 \geq \xi_1 - \xi_2 = x, \text{ and}$$
$$\xi_1 + \xi_2 \geq -\xi_1 + \xi_2 = -x.$$

Thus $\xi_1 + \xi_2 \geq |x|$. Similarly $\eta_1 + \eta_2 \geq |y|$ and $\zeta_1 + \zeta_2 \geq |z|$

Hence $\xi_1 + \xi_2 + \eta_1 + \eta_2 + \zeta_1 + \zeta_2 \geq |x| + |y| + |z| \geq |\hat{x}| + |\hat{y}| + |\hat{z}|$

$$= \hat{x}_+ + \hat{x}_- + \hat{y}_+ + \hat{y}_- + \hat{z}_+ + \hat{z}_-$$

And it is clear that $(\hat{x}_+, \hat{x}_-, \hat{y}_+, \hat{y}_-, \hat{z}_+, \hat{z}_-)$ is feasible for (LP). So we have shown that $(\hat{x}_+, \hat{x}_-, \hat{y}_+, \hat{y}_-, \hat{z}_+, \hat{z}_-)$ is optimal for (LP).

Next, suppose that $(\hat{\xi}_1, \hat{\xi}_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\zeta}_1, \hat{\zeta}_2)$ is optimal for (CLP). Define $(\hat{x}, \hat{y}, \hat{z})$ by $\hat{x} := \hat{\xi}_1 - \hat{\xi}_2$, $\hat{y} := \hat{\eta}_1 - \hat{\eta}_2$, $\hat{z} := \hat{\zeta}_1 - \hat{\zeta}_2$. Then $(\hat{x}, \hat{y}, \hat{z})$ is clearly feasible for (P). We claim that it is also optimal for (P). To this end, let (x, y, z) be feasible for (P). Then $(x_+, x_-, y_+, y_-, z_+, z_-)$ is feasible for (CLP). We have

$$
\begin{aligned}
|x|+|y|+|z| &= x_+ + x_- + y_+ + y_- + z_+ + z_- \\
&\geq \hat{\xi}_1 + \hat{\xi}_2 + \hat{\eta}_1 + \hat{\eta}_2 + \hat{\zeta}_1 + \hat{\zeta}_2 \\
&= |\hat{x}| + |\hat{y}| + |\hat{z}|.
\end{aligned}
$$

Hence $(\hat{x}, \hat{y}, \hat{z})$ is optimal for (P).

(Alterative solution: We can also give another linear programming reformulation of (P) by following the method outlined in Example 3.7, namely :

$$
(LP'): \begin{cases}
\text{minimize} & \xi + \eta + \zeta \\
\text{subject to} & \xi \geq x, \quad \eta \geq y, \quad \zeta \geq z, \\
& \xi \geq -x, \quad \eta \geq -y, \quad \zeta \geq -z, \\
& x + 2y = 1 \\
& x + z = 1
\end{cases}
$$

First of all, suppose that $(\hat{\xi}, \hat{\eta}, \hat{\zeta}, \hat{x}, \hat{y}, \hat{z})$ is optimal for (LP'). Then we claim that $(\hat{x}, \hat{y}, \hat{z})$ is optimal for (P). It is clear that $(|\hat{x}|, |\hat{y}|, |\hat{z}|, \hat{x}, \hat{y}, \hat{z})$ is feasible for (LP'). So $\hat{\xi} + \hat{\eta} + \hat{\zeta} \leq |\hat{x}| + |\hat{y}| + |\hat{z}|$ — (*). $(\hat{\xi}, \hat{\eta}, \hat{\zeta}, \hat{x}, \hat{y}, \hat{z})$ is feasible for (LP') and so $\left\{\begin{array}{l}\hat{\xi} \geq \hat{x} \\ \hat{\xi} \geq -\hat{x}\end{array}\right\}$, which gives $\hat{\xi} \geq |\hat{x}|$. Similarly $\hat{\eta} \geq |\hat{y}|$ and $\hat{\zeta} \geq |\hat{z}|$. Hence $\hat{\xi} + \hat{\eta} + \hat{\zeta} \geq |\hat{x}| + |\hat{y}| + |\hat{z}|$ — (**)

(*) and (**) together yield $\hat{\xi} + \hat{\eta} + \hat{\zeta} = |\hat{x}| + |\hat{y}| + |\hat{z}|$.

Let (x, y, z) be feasible for (P). Then $(|x|, |y|, |z|, x, y, z)$ is feasible for (LP'). Hence we obtain

$|x|+|y|+|z| \geq \hat{\xi}+\hat{\eta}+\hat{\zeta} = |\hat{x}|+|\hat{y}|+|\hat{z}|$. As $(\hat{x},\hat{y},\hat{z})$ is feasible for (P), we conclude that $(\hat{x},\hat{y},\hat{z})$ is optimal for (P).

Next, suppose that $(\hat{x},\hat{y},\hat{z})$ is optimal for (P). We will show that $(|\hat{x}|,|\hat{y}|,|\hat{z}|,\hat{x},\hat{y},\hat{z})$ is optimal for (LP'). Clearly $(|\hat{x}|,|\hat{y}|,|\hat{z}|,\hat{x},\hat{y},\hat{z})$ is feasible for (LP'). Suppose that (ξ,η,ζ,x,y,z) is feasible for (LP'). Then (x,y,z) is feasible for (P). Also $\xi \geq |x|$, $\eta \geq |y|$, $\zeta \geq |z|$. Thus $\xi+\eta+\zeta \geq |x|+|y|+|z| \geq |\hat{x}|+|\hat{y}|+|\hat{z}|$.

Hence $(|\hat{x}|,|\hat{y}|,|\hat{z}|,\hat{x},\hat{y},\hat{z})$ is optimal for (LP').)

Exercise 3.8

Let A denote the number of units of product A manufactured per day,
B denote the number of units of product B manufactured per day,
C denote the number of units of product C manufactured per day.

The profit made in a day is then $12A + 9B + 8C$.

The constraint in the department of Cutting is

$$\frac{A}{2000} + \frac{B}{1600} + \frac{C}{1100} \leq 8,$$

while the constraint in the department of Pressing is

$$\frac{A}{1000} + \frac{B}{1500} + \frac{C}{2400} \leq 8.$$

Also $A, B, C \geqslant 0$.

Hence we arrive at the following linear programming problem:

$$\left\{\begin{array}{ll} \text{Maximize} & 12A + 9B + 8C \\ \text{subject to} & \dfrac{A}{2000} + \dfrac{B}{1600} + \dfrac{C}{1100} \leq 8, \\ & \dfrac{A}{1000} + \dfrac{B}{1500} + \dfrac{C}{2400} \leq 8, \\ & A \geqslant 0, \\ & B \geqslant 0, \\ & C \geqslant 0. \end{array}\right.$$

Exercise 3.9

Let A = number of hectoliters of Apple cider produced in a week,

P = number of hectoliters of Pear cider produced in a week,

M = number of hectoliters of Mixed cider produced in a week,

S = number of hectoliters of Standard cider produced in a week.

Then the profit in a week is $196A + 210P + 280M + 442S$.

The production constraint is $1.6A + 1.8P + 3.2M + 5.4S \leq 80$,

while the packaging constraint is $1.2A + 1.2P + 1.2M + 1.8S \leq 40$.

The volume constraints are:

$$\begin{cases} \dfrac{A}{A+P+M+S} \geq \dfrac{20}{100} = \dfrac{1}{5}, & \text{and} \\[2ex] \dfrac{P}{A+P+M+S} \leq \dfrac{30}{100} = \dfrac{3}{10}, & \end{cases}$$

i.e.

$$\begin{cases} -\frac{4}{5}A + \frac{1}{5}P + \frac{1}{5}M + \frac{1}{5}S \leq 0, & \text{and} \\ -\frac{3}{10}A + \frac{7}{10}P - \frac{3}{10}M - \frac{3}{10}S \leq 0 & \end{cases}$$

Also, $A, P, M, S \geq 0$. Hence we arrive at the following linear programming problem:

$$\begin{cases} \text{Maximize} & 196A + 210P + 280M + 442S, \\ \text{subject to} & 1.6A + 1.8P + 3.2M + 5.4S \leq 80, \\ & 1.2A + 1.2P + 1.2M + 1.8S \leq 40, \\ & -0.8A + 0.2P + 0.2M + 0.2S \leq 0, \\ & -0.3A + 0.7P - 0.3M - 0.3S \leq 0, \\ & A \geq 0, \\ & P \geq 0, \\ & M \geq 0, \\ & S \geq 0. \end{cases}$$

Exercise 3.10

Let $S := \{1, 2, 3, \ldots, 12\}$ (set of stations), while
$P := \{(i,j) : i, j \in S \text{ and } i \neq j\}$ (set of distinct station pairs).

We have been given:
the constants p_i, $i \in S$,
the constants q_i, $i \in S$,
the constants r_{ij}, for $(i,j) \in P$.

We introduce the unknowns x_{ij} for $(i,j) \in P$, which will be variables in the formulation of the optimization problem.

First of all we have the constraints that $x_{ij} \geq 0$ for all $(i,j) \in P$.

The demand of "consistency" gives in addition the following constraints:

$$\sum_{i \in S} x_{ij} = q_j \quad \text{for all } j \in S$$
$$\sum_{j \in S} x_{ij} = p_i \quad \text{for all } i \in S.$$

So we arrive at the following optimization problem:

$$\left\{ \begin{array}{ll} \text{minimize} & \max_{(i,j) \in P} |x_{ij} - r_{ij}| \\ \text{subject to} & \sum_{i \in S} x_{ij} = q_j \text{ for all } j \in S, \\ & \sum_{j \in S} x_{ij} = p_i \text{ for all } i \in S, \\ & x_{ij} \geq 0 \text{ for all } (i,j) \in P. \end{array} \right.$$

This can be rewritten as a linear programming problem by introducing a new variable ω as follows:

$$\left\{ \begin{array}{ll} \text{minimize} & \omega \\ \text{subject to} & \omega \geq x_{ij} - r_{ij} \quad \text{for } (i,j) \in P, \\ & \omega \geq -(x_{ij} - r_{ij}) \quad \text{for } (i,j) \in P, \\ & \sum_i x_{ij} = q_j \quad \text{for } j \in S, \\ & \sum_j x_{ij} = p_i \quad \text{for } i \in S, \\ & x_{ij} \geq 0 \quad \text{for } (i,j) \in P. \end{array} \right.$$

Exercise 3.11

We introduce the following variables for $j=1,2,3$:

x_j = number of tonnes of the product manufactured in month j with normal working time,

y_j = number of tonnes of the product manufactured in month j with overtime;

z_j = number of tonnes of the product delivered to the customer at the end of month j,

s_j = number of tonnes of the product stored during the month j, and

u_j = number of tonnes of the product owed to the customer at the beginning of month j.

The cost in month j is

$$c x_j + d y_j + s s_j + f u_j$$

So the total cost is $\sum_{j=1}^{3} c x_j + d y_j + s s_j + f u_j$, and this should be minimized.

In month 1, $s_1 = 0$ (storage initially empty),
$u_1 = 0$ (nothing owed at beginning of month 1).

At the beginning of month 2, amount owed is $u_2 = q_1 - z_1$.

In month 2, amount stored is $s_2 = x_1 + y_1 - z_1$.

At the beginning of month 3, amount owed is $u_3 = q_2 - z_2 + u_2$

In month 3, amount stored is $s_3 = x_2 + y_2 - z_2 + s_2$.

At the end of month 3, amount delivered is $z_3 = q_3 + u_3$.

Also, the storage at the end of month 3 is empty, and so everything stored in month 3 together with everything produced in month 3 is actually delivered, i.e.,

$$x_3 + y_3 + s_3 = z_3$$

Also all variables are ≥ 0, and $x_j \leq a$, $y_j \leq b$ for $j=1,2,3$.

So we arrive at the following linear programming problem:

$$
\begin{cases}
\text{minimize} & \sum_{j=1}^{3} c x_j + d y_j + s s_j + f u_j \\
\text{subject to} & s_1 = 0, \\
& u_1 = 0, \\
& u_2 = q_1 - z_1, \\
& s_2 = x_1 + y_1 - z_1, \\
& u_3 = q_2 - z_2 + u_2, \\
& s_3 = x_2 + y_2 - z_2 + s_2, \\
& z_3 = q_3 + u_3, \\
& x_3 + y_3 + s_3 = z_3, \\
& x_1 \leq a, \quad y_1 \leq b, \\
& x_2 \leq a, \quad y_2 \leq b, \\
& x_3 \leq a, \quad y_3 \leq b, \\
& x_1, x_2, x_3 \geq 0, \\
& y_1, y_2, y_3 \geq 0, \\
& z_1, z_2, z_3 \geq 0, \\
& u_1, u_2, u_3 \geq 0, \\
& s_1, s_2, s_3 \geq 0.
\end{cases}
$$

Exercise 3.12

First of all we note that $\mathbb{P}$ is not empty, since $x = 0$ is a point in $\mathbb{P}$. Indeed $a_i^T x = 0 \leq b_i$ for all i. In fact $x = 0$ does not touch any of the walls, since $0 \notin P_i$ for all i. Indeed $a_i^T x = 0 < b_i$ for all i. So there is room inside $\mathbb{P}$ for at least a small ball (and hence a sphere) to be contained in $\mathbb{P}$.

Let z denote the center of the sought sphere, and let r denote its radius. Then z lies in $\mathbb{P}$ and so $a_i^T z \leq b_i$ for all i. Then the distance of z to P_i is given by
$$d(z, P_i) = \frac{|b_i - a_i^T z|}{\|a_i\|} \underset{(\because\ a_i^T z \leq b_i)}{=} \frac{b_i - a_i^T z}{\|a_i\|}.$$

The sphere with center z and radius r lies in $\mathbb{P}$ iff $r \leq d(z, P_i)$ for all i.

[(If): Let x belong to the sphere $\mathbb{S}$ with center z and radius r. Then $\|x - z\| = r$. We have
$$a_i^T x = a_i^T (x - z + z) = a_i^T (x - z) + a_i^T z \underset{\text{(Cauchy - Schwarz)}}{\leq} \|a_i\| r + a_i^T z \quad (1)$$
Now if $r \leq d(z, P_i)$, then $r \leq \dfrac{b_i - a_i^T z}{\|a_i\|}$, i.e., $\|a_i\| r + a_i^T z \leq b_i$. (2).
(1) and (2) give $a_i^T x \leq \|a_i\| r + a_i^T z \leq b_i$ for all i, and so $x \in \mathbb{P}$. So the sphere $\mathbb{S}$ lies in $\mathbb{P}$.

(Only if): The point $z + \dfrac{r}{\|a_i\|} a_i$ belongs to the sphere $\mathbb{S}$ with center z and radius r. If $\mathbb{S}$ lies in $\mathbb{P}$, then $a_i^T\left(z + \dfrac{r\, a_i}{\|a_i\|}\right) \leq b_i$ i.e., $a_i^T z + r\|a_i\| \leq b_i$,

i.e., $\dfrac{b_i - a_i^T z}{\|a_i\|} \geq r$ i.e., $d(z, P_i) \geq r$. Since the choice of i was arbitrary, this happens for all i.]

Hence we obtain the following linear programming problem in the variables $z \in \mathbb{R}^3$ and $r \in \mathbb{R}$:

$$\begin{cases} \text{maximize} & r \\ \text{subject to} & b_i - a_i^T z \geqslant r \|a_i\|, \quad i = 1, \ldots, m, \\ & r \geqslant 0. \end{cases}$$