Exercise 5.4

Introducing the slack variables x_5 and x_6, we transform the inequality contraints to equality constraints, and obtain the following linear programming problem in the standard form:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b, \\ & && x \geq 0, \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 \end{bmatrix},$$

$$b = \begin{bmatrix} 8 \\ 4 \end{bmatrix},$$

$$c = \begin{bmatrix} -3 & 4 & -2 & 5 & 0 & 0 \end{bmatrix}^T.$$

We start with x_5 and x_6 as basic variables. Thus $\beta = (5, 6)$ and $\nu = (1, 2, 3, 4)$.

Then $A_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_\nu = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$.

The initial basic solution is $x_\beta = \bar{b}$, where $A_\beta \bar{b} = b$.

Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{b} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, i.e., $\bar{b} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$.

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$ i.e., $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and so $y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The reduced costs of the nonbasic variables are

$$r_\nu = c_\nu - A_\nu^T y = c_\nu - A_\nu^T 0 = c_\nu = \begin{bmatrix} -3 \\ 4 \\ -2 \\ 5 \end{bmatrix}.$$

Since $r_{\nu_1} = r_1 = -3 < 0$ and the smallest, we make x_1 a new basic variable.

We compute $\bar{a}_1$ from $A_\beta \bar{a}_1 = a_1$ i.e., $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and so $\bar{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then the new basic variable x_1 can increase up to

$$t^{max} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{i,1}} : \bar{a}_{i,1} > 0 \right\} = \min\left\{ \frac{8}{1}, \frac{4}{1} \right\} = \frac{4}{1} = \frac{\bar{b}_2}{\bar{a}_{2,1}}.$$

The minimizing index is $r = 2$, and hence $x_{\beta_2} = x_6$ leaves the set of basic variables, and x_1 takes its place

So $\beta = (5, 1)$ and $\nu = (6, 2, 3, 4)$.

Hence $A_\beta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $A_\nu = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

Solving $A_\beta \bar{b} = b$, i.e., $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \bar{b} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, we obtain $\bar{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$ i.e., $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$, and so $y = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

The reduced costs of the nonbasic variables are

$$r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 0 \\ 4 \\ -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -2 \\ 5 \end{bmatrix} - \begin{bmatrix} -3 \\ 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

Since $r_\nu \geq 0$, the current basic feasible solution is optimal, i.e., $x = [4 \quad 0 \quad 0 \quad 0 \quad 4 \quad 0]^T$ is optimal for the problem in the standard form, and $\hat{x} = [4 \quad 0 \quad 0 \quad 0]$ is optimal for the original linear programming problem with <u>inequality</u> constraints

Now suppose that $c = [-3 \quad 4 \quad -2 \quad 2 \quad 0 \quad 0]^T$.

If we start from the final solution above, then we still have

$\beta = (5, 1)$,

$\nu = (6, 2, 3, 4)$

$A_\beta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A_\nu = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

$\bar{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$.

But the reduced costs of the nonbasic variables are:

$r_\nu = c_\nu - A_\nu^T y$, i.e.,

$$r_\nu = \begin{bmatrix} 0 \\ 4 \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Since $r_{\nu_4} = r_4 = -1 < 0$ and the smallest, x_4 becomes a new basic variable.

We compute $\bar{a}_4$ using $A_\beta \bar{a}_4 = a_4$ i.e., $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \bar{a}_4 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, i.e., $\bar{a}_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Since $\bar{a}_4 \leq 0$, x_4 can increase without constraint, and the value of the objective function goes to $-\infty$.

So the problem does not have an optimal solution, and the algorithm terminates.

(Remark: If we set $x_4 = t$ and let t increase from 0, while the other nonbasic variables stay at 0, then the objective function value changes as follows:

$$z = \bar{z} + r_4 t = c_\beta^T \bar{b} + r_4 t = \begin{bmatrix} 0 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} + (-1) t$$

$$= -12 - t.$$

The basic variables change according to

$$x_\beta = \bar{b} - t\bar{a}_4 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - t \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4+t \end{bmatrix}.$$

So $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} 4+t \\ 0 \\ 0 \\ t \\ 4 \\ 0 \end{bmatrix}$.

Since $Ax(t) = b$ and $x(t) \geq 0$ for all $t \geq 0$, $x(t)$ is a feasible solution for $t \geq 0$.

We have $c^T x(t) = -12 - t \rightarrow -\infty$ when $t \rightarrow +\infty$.)

Exercise 5.5.

The problem (LP) is in standard form:

$$(LP): \begin{cases} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{cases}$$

where $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \end{bmatrix}$,

$b = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$,

$c = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T$

The natural initial basic solution is obtained by taking x_5 and x_6 as basic variables. So $\beta = (5, 6)$ and $\nu = (1, 2, 3, 4)$.

Then $A_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_\nu = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$.

The values of the basic variables in the basic solution is given by $x_\beta = \bar{b}$, where $A_\beta \bar{b} = b$, i.e., $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{b} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$, and so $\bar{b} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$.

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$ i.e., $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and so $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The reduced costs for the nonbasic variables is given by

$$r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ -7 \\ -9 \end{bmatrix}.$$

Since $r_{\nu_4} = r_4 = -9$ is < 0 and the smallest, we let x_4 become a new basic variable.

We compute $\bar{a}_4$ using $A_\beta \bar{a}_4 = a_4$, i.e., $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{a}_4 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$,

and so $\bar{a}_4 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Then x_4 can increase up to

$$t_{max} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{i4}} : \bar{a}_{i4} > 0 \right\} = \min \left\{ \frac{10}{4}, \frac{12}{5} \right\} = \frac{12}{5} = \frac{\bar{b}_2}{\bar{a}_{24}}.$$

The minimizing index is $i = 2$ and hence $x_{\beta_2} = x_6$ leaves the set of basic variables, and x_4 takes its place.

Hence now $\beta = (5, 4)$ and $\nu = (1, 2, 3, 6)$.

So $A_\beta = \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix}$ and $A_\nu = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix}$.

The values of the basic variables in the basic solution is given by $x_\beta = \bar{b}$, where $A_\beta \bar{b} = b$, i.e.,

$\begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix} \bar{b} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$, and so $\bar{b} = \begin{bmatrix} 2/5 \\ 12/5 \end{bmatrix}$.

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$, i.e., $\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, i.e., $y = \begin{bmatrix} 1 \\ -4/5 \end{bmatrix}$.

The reduced costs for the non basic variables are given by

$$r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{-4}{5} \end{bmatrix} = \begin{bmatrix} -1 + \frac{8}{5} \\ -2 + \frac{12}{5} \\ -3 + \frac{16}{5} \\ 1 + \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \\ 1/5 \\ 9/5 \end{bmatrix}.$$

Since $r_\nu \geq 0$, the current basic feasible solution is optimal, i.e.,

$\hat{x} = [0 \quad 0 \quad 0 \quad 12/5 \quad 2/5 \quad 0]^T$ is optimal.

The optimal cost is

$c^T \hat{x} = \frac{2}{5} > 0$.

If $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ is a feasible solution to the original problem, then $x = (\xi_1, \xi_2, \xi_3, \xi_4, 0, 0)$ is a feasible solution to (LP). But the cost of (LP) with x is $0 < 2/5$, contradicting the optimality of $\hat{x}$ for (LP).

So the original problem has no feasible solution.

Exercise 5.6

The problem is in standard form: $\begin{cases} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{cases}$

where $A = \begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$,

$b = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$,

$c = \begin{bmatrix} 4 & 3 & 2 & 3 & 4 \end{bmatrix}^T$.

We start with $\beta = (1, 5)$ and $\nu = (2, 3, 4)$. Then

$A_\beta = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, $A_\nu = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

The values of the basic variables in the basic solution are given by $x_\beta = \bar{b}$, where $A_\beta \bar{b} = b$, i.e., $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \bar{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and so $\bar{b} = \begin{bmatrix} 5/4 \\ 3/4 \end{bmatrix}$.

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$ i.e., $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} y = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and so $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The reduced costs for the nonbasic variables is given by $r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$

Since $r_{\nu_2} = r_3 = -2$ is < 0 and the smallest, we let x_3 become a new basic variable.

We compute $\bar{a}_3$ using $A_\beta \bar{a}_3 = a_3$ i.e., $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \bar{a}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and so $\bar{a}_3 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

Then x_3 can increase upto $t_{max} = \min\limits_i \left\{ \dfrac{\bar{b}_i}{\bar{a}_{i,3}} : \bar{a}_{i,3} > 0 \right\}$

$$= \min \left\{ \frac{5/4}{1/2}, \frac{3/4}{1/2} \right\} = \frac{3}{2} = \frac{\bar{b}_2}{\bar{a}_{2,3}}$$

The minimizing index is $i = 2$ and hence $x_{\beta_2} = x_5$ leaves the set of basic variables, and x_3 takes its place

Hence now $\beta = (1, 3)$ and $\nu = (2, 5, 4)$. Then

$A_\beta = \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix}$ and $A_\nu = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 3 \end{bmatrix}$.

The values of the basic variables in the basic solution are given by $x_\beta = \bar{b}$, where $A_\beta \bar{b} = b$ i.e., $\begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix} \bar{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

and so $\bar{b} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$.

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$, i.e., $\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} y = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and so $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The reduced costs for the nonbasic variables are given by

$$r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 0 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}.$$

Since $r_\nu \geq 0$, the current basic feasible solution is optimal, i.e., $\hat{x} = [\,1/2 \quad 0 \quad 3/2 \quad 0 \quad 0\,]^T$ is optimal.

Since $r_{\nu_1} = r_2 = 0$, we can let x_2 become a new basic variable and let it increase from 0, without changing the value of the objective function.

Then we need to compute $\bar{a}_2$ using $A_\beta \bar{a}_2 = a_2$, i.e., $\begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix} \bar{a}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and so $\bar{a}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

The largest x_2 can get is $t_{max} = \min_i \left\{ \dfrac{\bar{b}_i}{\bar{a}_{i,2}} : \bar{a}_{i,2} > 0 \right\}$

$$= \min_i \left\{ \frac{1/2}{1/2}, \frac{3/2}{1/2} \right\} = 1.$$

The minimizing index is $i = 1$, and so $x_{\beta_1} = x_1$ leaves the set of basic variables, and x_2 replaces it. So $\beta = (2, 3)$ and $\nu = (1, 5, 4)$.

Then $A_\beta = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ and $A_\nu = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 3 \end{bmatrix}$

The values of the basic variables is given by $\bar{b}$, where $A_\beta \bar{b} = b$ i.e., $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \bar{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, and so $\bar{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So the new basic solution is $\hat{\hat{x}} = [0 \quad 1 \quad 1 \quad 0 \quad 0]^T$

(As a check, we note that the cost is

$$c^T\hat{\hat{x}} = 4\cdot 0 + 3\cdot 1 + 2\cdot 1 + 3\cdot 0 + 4\cdot 0 = 3+2 = 5,$$

which is the same as

$$c^T\hat{x} = 4\cdot 1/2 + 3\cdot 0 + 2\cdot \tfrac{3}{2} + 3\cdot 0 + 4\cdot 0 = 2+3 = 5.$$

Alternately, we can calculate the simplex multipliers vector y using $A_B^T y = c_B$ i.e., $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and so $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and next calculate the reduced costs for the nonbasic variables:

$$r_D = c_D - A_D^T y = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}.$$

Since $r_D \geq 0$, this basic solution is optimal.)

Exercise 5.7.

Introducing the extra variables x_4, x_5, x_6, the problem can be rewritten in the standard form as follows:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b, \\ & && x \geq 0, \end{aligned}$$

where $c = \begin{bmatrix} 1 & 5 & 2 & 0 & 0 & 0 \end{bmatrix}^T$,

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right],$$

$$b = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

The given starting solution is the basic solution with x_1, x_2, x_3 as the basic variables, that is, $\beta = (1, 2, 3)$ and $\nu = (4, 5, 6)$. Indeed we then have that the corresponding basic matrix is $A_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

and the values of the basic variables in the basic solution are given by $x_\beta = \bar{b}$, where $\bar{b}$ is obtained by solving $A_\beta \bar{b} = b$, i.e., $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \bar{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, and so

$$x_\beta = \bar{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$ i.e., $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$, and so $y = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

The reduced costs for the nonbasic variables are given by $r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - (-I) \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

Since $r_{\nu_2} = r_5 = -1$ is < 0 and the smallest, we make x_5 a new basic variable.

Then we calculate $\bar{a}_5$ using $A_\beta \bar{a}_5 = a_5$ i.e.,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \bar{a}_5 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and so} \quad \bar{a}_5 = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

Then x_5 can increase upto $t_{max} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{i,5}} \; ; \; \bar{a}_{i,5} > 0 \right\}$

$$= \frac{\bar{b}_2}{\bar{a}_{2,5}} = \frac{1}{1/2} = 2.$$

The minimizing index is $i = 2$ and hence $x_{\beta_2} = x_2$ leaves the set of basic variables, and x_5 takes its place. Hence now $\beta = (1, 5, 3)$ and $\nu = (4, 2, 6)$. Then

$$A_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_\nu = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

The values of the basic variables is given by $\bar{b}$, where $A_\beta \bar{b} = b$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \bar{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \text{and so} \quad \bar{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$, i.e.,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and so} \quad y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

The reduced costs for the nonbasic variables is given by

$$r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Since $r_\nu \geq 0$, the current basic solution is optimal. Hence $x_1 = 2$, $x_2 = 0$, $x_3 = 2$ is an optimal solution to the original problem. The optimal cost is given by $x_1 + 5x_2 + 2x_3 = 2 + 5 \cdot 0 + 2 \cdot 2 = 2 + 4 = 6.$

Exercise 5.8

The problem can be rewritten in the standard form

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b, \\ &&& x \geq 0, \end{aligned}$$

where $c = [-1 \quad -1 \quad -2 \quad 0 \quad 0]$,

$$A = \left[\begin{array}{ccc|cc} 1 & -1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 1 \end{array}\right],$$

$$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

by introducing the extra variables x_4 and x_5.
We start with the slack variables x_4 and x_5 as the basic variables. Thus $\beta = (4, 5)$ and $\nu = (1, 2, 3)$.
Then $A_\beta = I$ and $A_\nu = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

The values of the basic variables in the basic solutio are given by $x_\beta = \bar{b}$, where $A_\beta \bar{b} = b$, i.e., $I \bar{b} = b$, and so $\bar{b} = b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$ i.e, $Iy = c_\beta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and so $y = 0$.

The reduced costs for the nonbasic variables is given by
$r_\nu = c_\nu - A_\nu^T y = c_\nu - A_\nu^T 0 = c_\nu = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$.

Since $r_{\nu_3} = r_3 = -2 < 0$ and the smallest, we let x_3 become a new basic variable.
We compute $\bar{a}_3$ using $A_\beta \bar{a}_3 = a_3$ i.e., $I \bar{a}_3 = a_3$, and so $\bar{a}_3 = a_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Then x_3 can increase upto $t_{max} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{i,3}} : \bar{a}_{i,3} > 0 \right\}$

$$= \frac{\bar{b}_1}{\bar{a}_{1,3}} = \frac{1}{1} = 1.$$

The minimizing index is $i = 1$, and hence $x_{\beta_1} = x_4$ leaves the set of basic variables, and x_3 takes its place.
Hence now $\beta = (3, 5)$ and $\nu = (1, 2, 4)$.
Then $A_\beta = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $A_\nu = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

The values of the basic variables in the basic solution are given by $x_\beta = \bar{b}$, where $A_\beta \bar{b} = b$, i.e., $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \bar{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and so $\bar{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$, i.e., $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} y = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, and so $y = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

The reduced costs for the nonbasic variables are given by
$$r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$
Since $r_{\nu_2} = r_2 = -3 < 0$ and the smallest, we let x_2 become a new basic variable.
We compute $\bar{a}_2$ using $A_\beta \bar{a}_2 = a_2$ i.e., $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \bar{a}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and so $\bar{a}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Since $\bar{a}_2 \leq 0$, a solution does not exist.

We let $x_2 = t$, where t increases from 0. Meanwhile the other nonbasic variables are still at 0. Then $x_\beta(t) = \bar{b} - t\bar{a}_2$. Thus
$$\begin{aligned} x_1(t) &= 0 \\ x_2(t) &= t \\ x_3(t) &= 1 + t \\ x_4(t) &= 0 \\ x_5(t) &= 2. \end{aligned}$$

In terms of the variables of the original problem, we have
$$\left.\begin{aligned} x_1(t) &= 0 \\ x_2(t) &= t \\ x_3(t) &= 1 + t \end{aligned}\right\}, \text{ i.e., } x(t) = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{=: x_0} + t \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{=: d}.$$

Then $$P\,x(t) = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ t \\ 1+t \end{bmatrix} = \begin{bmatrix} -t+1+t \\ t-1-t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} = b,$$

$$x(t) = \begin{bmatrix} 0 \\ t \\ 1+t \end{bmatrix} \geqslant 0 \quad \forall\, t \geqslant 0,$$

and so $x(t)$ is a feasible solution for each $t \geqslant 0$.

Furthermore, $$q^T x(t) = [1 \;\; 1 \;\; 2] \begin{bmatrix} 0 \\ t \\ 1+t \end{bmatrix} = t+2+2t = 2+3t \longrightarrow +\infty$$ as $t \longrightarrow +\infty$

Exercise 5.9

We have $\beta = (1, 3, 5)$ and $\nu = (2, 4, 6)$. Then

$$A_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} \qquad \left(\text{and } A_\nu = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}\right).$$

(A_β is invertible, and so it is indeed a basic matrix.)

The values of the basic variables in the basic solution are given by $x_\beta = \bar{b}$, where $\bar{b}$ is determined by $A_\beta \bar{b} = b$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} \bar{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \text{ and so } \bar{b} = \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} \geq 0.$$

So the corresponding solution is feasible.

So $x = [1 \;\; 0 \;\; 1/2 \;\; 0 \;\; 1 \;\; 0]^T$ is a basic feasible solution.

The simplex multipliers vector y is given by $A_\beta^T y = c_\beta$, i.e.,

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} y = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \text{ and so } y = \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}.$$

The reduced costs for the nonbasic variables are given by

$$r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ +3/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}.$$

Since $r_{\nu_2} = r_4 = -\frac{1}{2} < 0$, we let x_4 become a new basic variable.

We then compute $\bar{a}_4$ using $A_\beta \bar{a}_4 = a_4$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} \bar{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and so } \bar{a}_4 = \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}.$$

Then x_4 can be increased upto

$$t_{max} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{i,4}} \; ; \; \bar{a}_{i,4} > 0 \right\} = \frac{\bar{b}_1}{\bar{a}_{1,4}} = \frac{1}{1} = 1.$$

The minimizing index is $i = 1$, and so $x_{\beta_1} = x_1$ will leave the set of basic variables, and is replaced by x_4.

Thus $\beta = (4, 3, 5)$ and $\nu = (2, 1, 6)$. Then

$$A_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad A_\nu = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

The values of the basic variables in the basic solution are given by $x_\beta = \bar{b}$, where $A_\beta \bar{b} = b$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix} \bar{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \text{and so} \quad \bar{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(As expected, this is a basic feasible solution.)

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$, i.e.,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and so} \quad y = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

The reduced costs for the nonbasic variables are given by

$$r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 3/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}$$

Since $r_\nu \geq 0$, the current basic solution is optimal.

Thus

$x = [0 \ 0 \ 1 \ 1 \ 1 \ 0]^T$ is a basic feasible solution which is optimal.

Exercise 5.10

We have $c = [0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1]^T$ and let $G := [A \mid I]$.

First of all, it is feasible, since

$$G\begin{bmatrix} x \\ y \end{bmatrix} = Ax + Iy = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = b,$$

$$x = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix} \geqslant 0 \quad \text{and} \quad y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \geqslant 0.$$

Moreover, it is a basic solution with $\beta = (1, 2, 7)$.

The simplex multipliers vector is obtained by solving $G_\beta^T Y =$

i.e., $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^T y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, that is, $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

and so $y = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. We have $G_\nu = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$.

The reduced costs for the nonbasic variables are given by

$$r_\nu = c_\nu - G_\nu^T Y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

Since $r_\nu \geqslant 0$, the basic feasible solution is optimal.

There do not exist nonnegative scalars x_j such that $b = x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4$. For if they did, there would be a feasible solution to (LP) with $y = 0$, and the corresponding cost would be $0 + 0 + 0 = 0 < 1$, contradicting the optimality of $y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ shown above.