<u>Last time:</u>

$$\begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \leq 0 \\ & \vdots \\ & g_m(x) \leq 0 \end{cases}$$

$$\left.\begin{array}{l} x_0 \text{ local minimizer} \\ \quad\text{and} \\ x_0 \text{ regular point} \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} \exists\, y \in \mathbb{R}^m \text{ s.t. the following KKT-conditions hold:} & \\ \text{(KKT-1)} & \nabla f(x_0) + y_1 \nabla g_1(x_0) + \dots + y_m \nabla g_m(x_0) = 0 \\ \text{(KKT-2)} & \begin{cases} g_1(x_0) \leq 0 \\ \vdots \\ g_m(x_0) \leq 0 \end{cases} \\ \text{(KKT-3)} & \begin{cases} y_1 \geq 0 \\ \vdots \\ y_m \geq 0 \end{cases} \\ \text{(KKT-4)} & \begin{cases} y_1 g_1(x_0) = 0 \\ \vdots \\ y_m g_m(x_0) = 0 \end{cases} \end{array}\right.$$

Today we will see that for convex problems, the KKT-conditions are <u>sufficient</u> for an optimal solution!

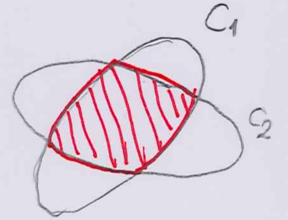
Recall that a problem $\left\{ \begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & x \in \mathcal{X} \end{array} \right\}$ is called a <u>convex problem</u>

if $\mathcal{X} \subset \mathbb{R}^n$ is a convex set and $f : \mathcal{X} \to \mathbb{R}$ is a convex function.

<u>Proposition</u>. $(P): \left\{ \begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \le 0 \\ & \vdots \\ & g_m(x) \le 0 \end{array} \right\}$ is a convex problem if $f, g_1, \dots, g_m$ are convex.

<u>Proof</u>. The set $C_i := \{ x \in \mathbb{R}^n : g_i(x) \le 0 \}$ is convex.

$\Big($ $\because$ $x, y \in C_i$, $t \in (0,1) \Rightarrow \begin{array}{l} g_i(x) \le 0 \\ g_i(y) \le 0 \end{array}$, and so
$$g_i((1-t)x + ty) \le \underbrace{(1-t)}_{\ge 0} \underbrace{g(x)}_{\le 0} + \underbrace{t}_{\ge 0} \underbrace{g(y)}_{\le 0} \le 0.$$
So $(1-t)x + ty \in C_i \Big)$.



Hence $C_1, \dots, C_m$ are convex.

But then $\mathcal{X} := C_1 \cap \dots \cap C_m$ is also convex.

$x, y \in \mathcal{X}$, $t \in (0,1)$.

Fix i. Then $x, y \in C_i$, $t \in (0,1) \overset{\because C_i \text{ is convex}}{\Longrightarrow} (1-t)x + ty \in C_i$

Choice of i was arbitrary. Thus $(1-t)x + ty \in C_1 \cap \dots C_m = \mathcal{X}$.

Finally $f : \mathcal{X} \to \mathbb{R}$ is convex (given). So (P) is a convex problem. $\square$

<u>Theorem</u> (<u>Sufficiency</u> of the KKT-conditions for <u>convex problems</u>).

Consider (P): $\left\{ \begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t} & g_1(x) \leq 0 \\ & \vdots \\ & g_m(x) \leq 0 \end{array} \right\}$, where $f, g_1, \ldots, g_m$ are convex functions.

Suppose that $\hat{x} \in \mathbb{R}^n$ and $\hat{y} \in \mathbb{R}^m$ satisfy:

(KKT-1) $\nabla f(\hat{x}) + \hat{y}_1 \nabla g_1(\hat{x}) + \ldots + \hat{y}_m \nabla g_m(\hat{x}) = 0$

(KKT-2) $\left\{ \begin{array}{l} g_1(\hat{x}) \leq 0 \\ \vdots \\ g_m(\hat{x}) \leq 0 \end{array} \right.$

(KKT-3) $\left\{ \begin{array}{l} \hat{y}_1 \geq 0 \\ \vdots \\ \hat{y}_m \geq 0 \end{array} \right.$

(KKT-4) $\left\{ \begin{array}{l} \hat{y}_1 g_1(\hat{x}) = 0 \\ \vdots \\ \hat{y}_m g_m(\hat{x}) = 0 \end{array} \right.$.

<u>Then</u> $\hat{x}$ is a global minimizer for (P).

Proof (KKT-2) $\Rightarrow$ $\hat{x}$ is feasible.

Consider the auxiliary function $L : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$L(x) = f(x) + \hat{y}_1 g_1(x) + \dots + \hat{y}_m g_m(x), \qquad x \in \mathbb{R}^n$$

Note that $\hat{y}_1, \dots, \hat{y}_m$ are fixed numbers (given), and x is the variable.

<u>Claim</u>: L is convex.

If $x, y \in \mathbb{R}^n$, $t \in (0,1)$, then:

$$L((1-t)x + ty) = \underbrace{f((1-t)x + ty)}_{\leq} + \sum_{i=1}^{m} \underbrace{\hat{y}_i}_{\geq 0} \underbrace{g_i((1-t)x + ty)}_{\leq}$$

$$\leq (1-t) f(x) + t f(y) + \sum_{i=1}^{m} \hat{y}_i \left((1-t) g_i(x) + t g_i(y) \right)$$

$$= (1-t) \left[f(x) + \sum_{i=1}^{m} \hat{y}_i g_i(x) \right] + t \left[f(y) + \sum_{i=1}^{m} \hat{y}_i g_i(y) \right]$$

$$= (1-t) L(x) + t L(y).$$

This proves the claim.

Let us now calculate $\nabla L(\hat{x})$.

$$\nabla L(\hat{x}) = \nabla f(\hat{x}) + \sum_{i=1}^{m} \hat{y}_i \nabla g_i(\hat{x}) \overset{\text{(KKT-1)}}{=} 0 \; !$$

L convex and $\nabla L(\hat{x}) = 0 \Longrightarrow \hat{x}$ is a global minimizer for L.

Let $x \in \mathcal{F}_e$. Then $\left\{\begin{array}{l} g_1(x) \leq 0 \\ \vdots \\ g_m(x) \leq 0 \end{array}\right\}$. We will show that $f(x) \geqslant f(\hat{x})$.

Know: $L(x) \geqslant L(\hat{x})$, i.e.,

$$f(x) \geqslant \boxed{f(x) + \underbrace{\sum_{i=1}^{m} \underbrace{\hat{y}_i}_{\geqslant 0} \underbrace{g_i(x)}_{\leqslant 0}}_{\leqslant 0}} \geqslant f(\hat{x}) + \underbrace{\sum_{i=1}^{m} \hat{y}_i \, g_i(\hat{x})}_{= 0 \text{ by (KKT-4)}}$$

So $f(x) \geqslant f(\hat{x})$ $\forall x \in \mathcal{F}_e$.

Hence $\hat{x}$ is an optimal solution for (P). Done! $\square$

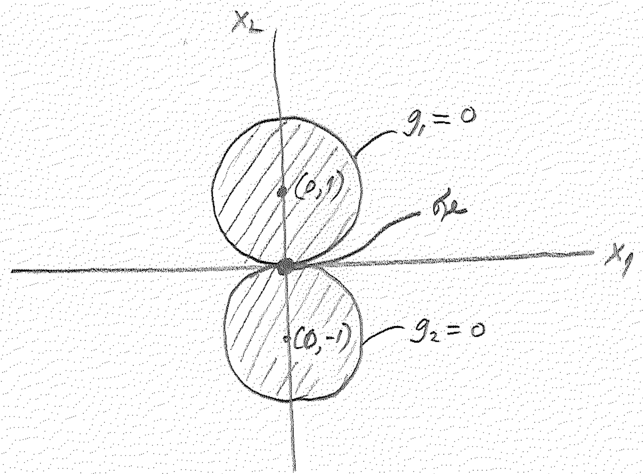
Are the conditions also necessary?

Example

$$(P):\begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \end{cases} \quad \text{where} \quad \begin{aligned} f(x_1, x_2) &:= x_1 \\ g_1(x_1, x_2) &:= x_1^2 + (x_2 - 1)^2 - 1 \\ g_2(x_1, x_2) &:= x_1^2 + (x_2 + 1)^2 - 1 . \end{aligned}$$

Then f, g_1, g_2 are convex. ($\because$ f is linear and the Hessians of g_1, g_2 are p.s.d.: $G_1(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is p.d, $G_2(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is p.d.)

So (P) is a convex problem.



$\mathcal{F} = \{(0,0)\}$.

So $(0, 0)$ is the (unique) global minimizer.

But we will now see that the KKT conditions are not satisfied with $\hat{x} := (0,0)$:

$$\nabla f(0) = [1 \quad 0]$$

$$\nabla g_1(0) = \left[2x_1 \quad 2(x_2 - 1)\right]\Big|_{(x_1,x_2)=(0,0)} = [0 \quad -2].$$

$$\nabla g_2(0) = \left[2x_1 \quad 2(x_2 + 1)\right]\Big|_{(x_1,x_2)=(0,0)} = [0 \quad 2].$$

So (KKT-1) becomes $\nabla f(0) + \hat{y}_1 \nabla g_1(0) + \hat{y}_2 \nabla g_2(0) = 0$ i.e.,

$$[1 \quad 0] + \hat{y}_1 [0 \quad -2] + \hat{y}_2 [0 \quad 2] = [1 \quad *] = [0 \quad 0],$$

a contradiction!

What went wrong? $(0,0)$ is not a regular point.

(Recall that $x \in \mathcal{F}$ is not a regular point if $\exists$ nonnegative v_i $(i \in I_a(x))$, not all zeros, s.t. $\sum_{i \in I_a(x)} v_i \nabla g_i(x) = 0$.)

In our case, $I_a(0,0) = \{1, 2\}$. With $v_1 := 1$, $v_2 := 1$, we have $v_1 \cdot \nabla g_1(0) + v_2 \cdot \nabla g_2(0) = 1 \cdot [0 \quad -2] + 1 \cdot [0 \quad 2] = 0$. So $(0,0)$ is not regular.

<u>Know</u>: (1)

$$\left.\begin{array}{l}\hat{x} \text{ global minimizer} \\ \quad\text{and} \\ \hat{x} \text{ regular}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\hat{x} \text{ satisfies the} \\ \text{KKT - conditions}\end{array}\right.$$

(2) For <u>convex</u> problems,

$$\left.\begin{array}{l}\hat{x} \text{ satisfies the} \\ \text{KKT- condition}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\hat{x} \text{ is a global} \\ \text{minimizer}\end{array}\right.$$

It would be good to have an 'iff' statement for convex problems.

One can show the following:

<u>Proposition</u>: If $g_1, \dots, g_m$ are convex and

$$\exists\, x_0 \text{ s.t. } \begin{array}{l} g_1(x_0) < 0 \\ \vdots \\ g_m(x_0) < 0 \end{array}$$

then every feasible point is regular for $\left\{\begin{array}{ll}\text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \leq 0 \\ & \vdots \\ & g_m(x) \leq 0.\end{array}\right.$

The problem $\left\{\begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \le 0 \\ & \vdots \\ & g_m(x) \le 0 \end{array}\right\}$ is called a <u>regular problem</u>

if there exists a $x_0 \in \mathbb{R}^n$ s.t. $\begin{cases} g_1(x_0) < 0 \\ \vdots \\ g_m(x_0) < 0 \end{cases}$.

<u>Theorem</u>. Suppose that (P): $\left\{\begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \le 0 \\ & \vdots \\ & g_m(x) \le 0 \end{array}\right\}$ is a regular problem,

and that $f, g_1, \dots, g_m$ are convex functions.

Then $\hat{x}$ is an optimal solution for (P)

iff

$\exists\ \hat{y} \in \mathbb{R}^m$ s.t. (KKT-1) $\nabla f(\hat{x}) + \hat{y}_1 \nabla g_1(\hat{x}) + \dots + \hat{y}_m \nabla g_m(\hat{x}) = 0$

(KKT-2) $\begin{cases} g_1(\hat{x}) \le 0 \\ \vdots \\ g_m(\hat{x}) \le 0 \end{cases}$

(KKT-3) $\begin{cases} \hat{y}_1 \ge 0 \\ \vdots \\ \hat{y}_m \ge 0 \end{cases}$

(KKT-4) $\begin{cases} \hat{y}_1 g_1(\hat{x}) = 0 \\ \vdots \\ \hat{y}_m g_m(\hat{x}) = 0 \end{cases}$.

<u>Example</u>

$$\begin{cases} \text{minimize} & x_1^2 - 2x_1 + x_2^2 + 1 \\ \text{s.t.} & x_1 + x_2 \leq 0 \\ & x_1^2 - 4 \leq 0 \end{cases}$$

$$\left.\begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \end{cases}\right\} \text{ where } \begin{aligned} f(x) &:= x_1^2 - 2x_1 + x_2^2 + 1 \\ g_1(x) &:= x_1 + x_2 \\ g_2(x) &:= x_1^2 - 4 \end{aligned}$$

g_1 is convex, since it is linear.

Hessians of f, g_2: $\nabla f(x) = [2x_1 - 2 \quad 2x_2]$ $\quad F(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ p.d.

$\nabla g_2(x) = [2x_1 \quad 0]$ $\quad G_2(x) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ p.s.d.

So f, g_2 are convex as well.

<u>Is the problem regular?</u> With $x_0 := (-1, 0)$, we have

$$g_1(x_0) = g_1(-1, 0) = -1 + 0 = -1 < 0$$

$$g_2(x_0) = g_2(-1, 0) = (-1)^2 - 4 = -3 < 0.$$

So the problem is regular.

$\Rightarrow$ KKT conditions are necessary and sufficient.

(KKT-1):

$$\nabla f(x) + y_1 \nabla g_1(x) + y_2 \nabla g_2(x) = 0$$

$$[2x_1 - 2 \quad 2x_2] + y_1 [1 \quad 1] + y_2 [2x_1 \quad 0] = 0$$

$$\begin{cases} 2x_1 - 2 + y_1 + 2y_2 x_1 = 0 \\ 2x_2 + y_1 = 0 \end{cases}$$

(KKT-2):

$$\begin{cases} x_1 + x_2 \leq 0 \\ x_1^2 - 4 \leq 0 \end{cases}$$

(KKT-3):

$$\begin{cases} y_1 \geq 0 \\ y_2 \geq 0 \end{cases}$$

(KKT-4):

$$y_1 (x_1 + x_2) = 0$$

$$y_2 (x_1^2 - 4) = 0$$

$\boxed{\begin{array}{l} y_1 > 0 \\ y_2 > 0 \end{array}}$

(KKT-4) : $\left.\begin{array}{l} x_1 + x_2 = 0 \\ x_1^2 - 4 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = +2 \text{ or } -2 \\ x_2 = -2 \text{ or } +2. \end{array}$ $(x_1, x_2) = (2, -2)$ or $(-2, 2)$

$\boxed{1^\circ}$ $(x_1, x_2) = (2, -2)$. (KKT-1) : $\left.\begin{array}{l} 4 - 2 + y_1 + 2y_2(2) = 0 \\ 2(-2) + y_1 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} y_1 = 4 \\ y_2 = -6/4 \ ↯ \end{array}$

(KKT-3) violated.

$\boxed{2^\circ}$ $(x_1, x_2) = (-2, 2)$. (KKT-1) : $\left.\begin{array}{l} -4 - 2 + y_1 + 2y_2(-2) = 0 \\ 2(2) + y_1 = 0 \end{array}\right\} \Rightarrow y_1 = -4 \ ↯$

(KKT-3) violated

$\boxed{\begin{array}{l} y_1 > 0 \\ y_2 = 0 \end{array}}$

(KKT-4) : $x_1 + x_2 = 0.$

(KKT-1) : $\left.\begin{array}{l} 2x_1 - 2 + y_1 = 0 \\ 2x_2 + y_1 = 0 \end{array}\right\}$

$x_1 = -x_2$

$$\begin{array}{r} -2x_2 - 2 + y_1 = 0 \\ 2x_2 \quad + y_1 = 0. \\ \hline -2 + 2y_1 = 0 \end{array} \Rightarrow \begin{array}{l} y_1 = 1 \\ x_2 = -\frac{1}{2} \\ x_1 = \frac{1}{2} \end{array}$$

$\frac{1}{2} + -\frac{1}{2} = 0$

$\left(\frac{1}{2}\right)^2 - 4 < 0.$ so (KKT-2) is satisfied

So with $x = \left(\frac{1}{2}, -\frac{1}{2}\right)$ and $y = (1, 0)$, all the KKT conditions are satisfied.

So $x = \left(\frac{1}{2}, -\frac{1}{2}\right)$ is a global minimizer.

Are there any others?

$\boxed{\begin{array}{l} y_1 = 0 \\ y_2 > 0 \end{array}}$

(KKT-4): $x_1^2 - 4 = 0 \Rightarrow x_1 = +2$ or -2

$\boxed{1^\circ}$ $x_1 = +2$. (KKT-1): $\left.\begin{array}{r} 4 - 2 + 0 + 4y_2 = 0 \\ 2x_2 + 0 = 0 \end{array}\right\} \Rightarrow y_2 = -1/2$ ↯. (KKT-3) is violated.

$\boxed{2^\circ}$ $x_1 = -2$. (KKT-1): $\cancel{4}\,{-4} - 2 + 0 - 4y_2 = 0 \Rightarrow y_2 = -6/4$ ↯ (KKT-3) is violated.

$\boxed{\begin{array}{l} y_1 = 0 \\ y_2 = 0 \end{array}}$

(KKT-1): $\left.\begin{array}{r} 2x_1 - 2 = 0 \\ 2x_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = 0 \end{array}$. But then $x_1 + x_2 = 1 + 0 = 1 \not\leq 0$.

So (KKT-2) is violated.

So there is an optimal solution, namely $\hat{x} = \left(\frac{1}{2}, -\frac{1}{2}\right)$,

and there are no others.

$f(x) = x_1^2 - 2x_1 + x_2^2 + 1 = x_1^2 - 2x_1 + 1 + x_2^2 = (x_1 - 1)^2 + x_2^2$.

Level curves $\{x \in \mathbb{R}^2 : f(x) = V\}$ will be circles.

<u>Feasible set:</u>