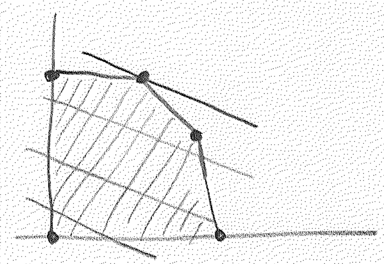
<u>Basic feasible solutions</u> $\equiv$ corner points.

In our graphical study of LP problems in $\mathbb{R}^2$, we had noticed that if there exists an optimal solution, then there is an optimal solution among the <u>corner points</u> of the feasible set.



Problem amounts to searching for an optimal solution among a <u>finite</u> number of points.

The same thing happens in $\mathbb{R}^n$!

Consider $(P): \begin{cases} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geqslant 0 \end{cases}$ $\quad A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ $\quad x \in \mathbb{R}^n$ variable

<u>Theorem</u> (Fundamental theorem of LP)

If there exists an optimal solution to (P), then there exists an optimal solution which is a <u>basic feasible solution</u> to (P)

"corner" points of $\mathcal{X}$

$\# \{ \text{basic feasible solutions to } (P) \} \leq \binom{n}{m} < +\infty$!

(1) $\mathcal{X} = \{ x \in \mathbb{R}^n \mid Ax = b \text{ and } x \geqslant 0 \}$ Feasible set.

(2) x feasible for (P) if $x \in \mathcal{X}$.

(3) $\hat{x}$ optimal solution for (P) if $\hat{x} \in \mathcal{X}$ and $\forall x \in \mathcal{X}$, $c^T \hat{x} \leq c^T x$.

$$\boxed{\underline{\text{Assumption:}} \quad \operatorname{rank} A = m}$$

<u>Remarks:</u> (1) $\Leftrightarrow$ Rows of A are linearly independent

(2) Not a severe assumption.

If $b \notin$ range of A, then $\mathcal{F} = \emptyset$ and the problem has no solution.

If $b \in$ range of A and rows aren't independent, then we can delete some of the rows of A (and corresponding entries of b) to ensure that the left over rows are linearly independent, without changing the feasible set.

(3) $A \in \mathbb{R}^{m \times n}$ $\quad \operatorname{rank} A \leq n$. So $m \leq n$ $\quad A = \square$ or $\boxed{\quad\quad\quad}$ (not tall)

A is invertible.

$\Rightarrow \mathcal{F}$ has just one element and so (P) is trivial!

So actually it makes sense to assume $\operatorname{rank} A = \boxed{m < n}$.

(4) $Ax = b$ always has a solution $x \in \mathbb{R}^n$, since the columns of A span $\mathbb{R}^m$.

But this x may not be feasible, since it may not be ≥ 0.

<u>Question</u>: What are basic feasible solutions?

<u>Answer</u>: They are special solutions to $Ax = b$.

$$Ax = b$$

$$\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b$$

$$x_1 \begin{bmatrix} a_1 \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_n \end{bmatrix} = b. \qquad (*)$$

Any <u>m independent</u> columns of A form a basis for $\mathbb{R}^m$.

Suppose we pick $a_{\beta_1}, \dots, a_{\beta_m}$ which are independent.

Then we can always find $x_{\beta_1}, \dots, x_{\beta_m}$ s.t

$$x_{\beta_1} \begin{bmatrix} a_1 \end{bmatrix} + \cdots + x_{\beta_m} \begin{bmatrix} a_{\beta_m} \end{bmatrix} = b \quad \text{(and these are unique).}$$

So we can solve $(*)$ where the $x_i = 0$ if i is not one of $\beta_1, \dots, \beta_m$ and $x_i = x_{\beta_k}$ if $i = \beta_k$.

This x is called a <u>basic solution</u>. If $x \geq 0$, it is a basic feasible soln.

$\beta = (\beta_1, \dots, \beta_m) \in \{1, \dots, n\}^m$ is called a <u>basic tuple</u> if $a_{\beta_1}, \dots, a_{\beta_m}$ are linearly independent.

$A_\beta := \left[a_{\beta_1} \cdots a_{\beta_m} \right] \in \mathbb{R}^{m \times m}$ is called a <u>basic matrix</u>.

<u>Example</u> (furniture production planning)

$$A = \begin{array}{cccc} \left[\begin{matrix} 1 \\ 2 \end{matrix} \right. & \begin{matrix} 1 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} & \left. \begin{matrix} 0 \\ 1 \end{matrix} \right] \\ a_1 & a_2 & a_3 & a_4 \end{array} \qquad b = \begin{bmatrix} 200 \\ 300 \end{bmatrix}.$$

(i) $\beta = (4, 1)$ is a basic tuple since a_1, a_4 are linearly independent.

$$A_\beta = \begin{array}{cc} \left[\begin{matrix} 0 \\ 1 \end{matrix} \right. & \left. \begin{matrix} 1 \\ 2 \end{matrix} \right] \\ a_4 & a_1 \end{array}$$

(ii) $\beta = (1, 2)$ is also a basic tuple. $A_\beta = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

$\beta = (1, 3)$ is not basic, since a_1, a_3 are linearly dependent.

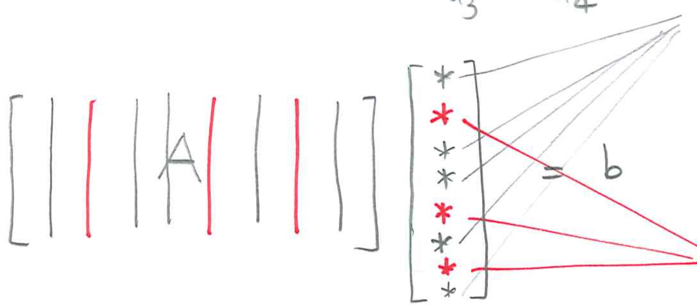
Left over indices $\nu = (\nu_1, \dots, \nu_{n-m})$: non-basic tuple.

Left over columns of A are collected to form $A_\nu = [\,a_{\nu_1} \cdots a_{\nu_{n-m}}\,] \in \mathbb{R}^{m \times (n-m)}$.

<u>Example</u> (continued)

(i) $\nu = (2, 3)$ $\quad A_\nu = \begin{array}{cc} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ a_2 \quad a_3 \end{array}$.

(ii) $\nu = (3, 4)$ $\quad A_\nu = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ a_3 \quad a_4 \end{array}$.



$x_\beta = \begin{bmatrix} x_{\beta_1} \\ \vdots \\ x_{\beta_m} \end{bmatrix}$ basic variables.

$x_\nu = \begin{bmatrix} x_{\nu_1} \\ \vdots \\ x_{\nu_{n-m}} \end{bmatrix}$ non-basic variables.

$$\boxed{Ax = b}$$

$$x_1 \boxed{a_1} + \dots + x_n \boxed{a_n} = b$$

$$x_{\beta_1} \boxed{a_1} + \dots + x_{\beta_m} \boxed{a_{\beta_m}} + x_{\nu_1} \boxed{a_{\nu_1}} + \dots + x_{\nu_{n-m}} \boxed{a_{\nu_{n-m}}} = b$$

$$\boxed{A_\beta x_\beta + A_\nu x_\nu = b.}$$

Then solve for x_β: $A_\beta x_\beta = b$.

$$x_\beta = A_\beta^{-1} b.$$

Set $x_\nu = 0$.

Then the x formed from putting together x_β and $x_\nu\ (=0)$ satisfies

$$Ax = A_\beta x_\beta + A_\nu x_\nu = b + 0 = b.$$

This x is called <u>the basic solution (corresponding to β)</u>.

<u>Example</u> (continued)

(i) $\beta = (4, 1)$ $\quad A_\beta = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

$$A_\beta x_\beta = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_4 \\ x_1 \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \end{bmatrix} = b \Rightarrow \begin{array}{l} x_1 = 200 \\ x_4 = -100 \end{array}$$

$$x_\nu = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = \begin{bmatrix} 200 \\ 0 \\ 0 \\ -100 \end{bmatrix}$ is the basic solution corresponding to $\beta = (4, 1)$.

(ii) $\beta = (1, 2)$ $\qquad A_\beta = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

$$A_\beta x_\beta = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \end{bmatrix} = b. \qquad \begin{array}{l} x_1 = 100 \\ x_2 = 100 . \end{array}$$

$$x_\nu = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = \begin{bmatrix} 100 \\ 100 \\ 0 \\ 0 \end{bmatrix}$ is the basic solution corresponding to $\beta = (1, 2)$

If a basic solution is $\geqslant 0$, it is called a <u>basic feasible solution</u>.

In the example above, the x in (i) <u>is not</u> a basic feasible solution since $x_4 = -100 < 0$;

the x in (ii) <u>is</u> a basic feasible solution since $x = \begin{bmatrix} 100 \\ 100 \\ 0 \\ 0 \end{bmatrix} \geqslant 0$.

So now we know what is meant by a <u>basic feasible solution</u>

Recap: Let β be a basic tuple (corresponding columns of A are lin. indep.)

Solve for x_β: $A_\beta x_\beta = b$.

Set $x_\nu = 0$.

Put x_β, x_ν together to form x.

This x is a basic solution corresponding to β.

We have $Ax = b$.

But it may be that x is not $\geqslant 0$.

Ask: Is $x \geqslant 0$?

If Yes, then this x is a basic feasible solution. We have $Ax = b$ and $x \geqslant 0$.

If no, then this x, although a basic solution, is not a basic feasible solution.

No. of ways of choosing m columns from the n columns of A $= \binom{n}{m}$.

So no. of basic solutions is $\leq \binom{n}{m}$.

So no. of basic feasible solutions is $\leq \binom{n}{m} < +\infty$.

Recall the Fundamental Theorem of LP:

<u>Theorem</u>. If there exists an optimal solution to (P), then there is an optimal solution which is a basic feasible solution to (P).

So in principle, if we knew somehow that the problem has an optimal solution, then we could just calculate all basic feasible solutions, and find an optimal solution by just taking the one(s) giving least cost.

But this is not done in practice.

<u>Reason</u>: Even for small n, m, $\binom{n}{m}$ can be terribly large.

E.g. $n = 50$, $m = 5$ $\Rightarrow \binom{n}{m} = \binom{50}{5} = \dfrac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}118{,}760$.

So we need an efficient way of checking optimality among the basic feasible solutions.

Such an algorithm exists, and is called the <u>simplex method</u>

In the simplex method, we <u>don't</u> calculate <u>all</u> basic feasible solutions first.

Instead, we start with (any) one basic feasible solution.

We ask: is this optimal?

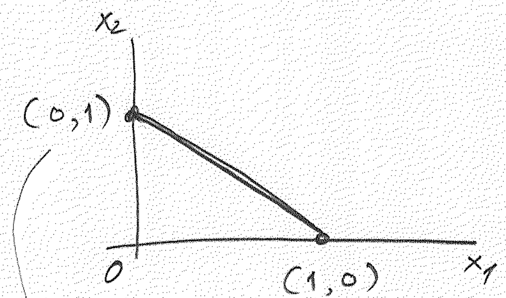
If yes: done.

If no: the algorithm moves to a new basic feasible solution with a lower cost.

Basic feasible solutions ≡ corner points of $\mathcal{X}$

(1) $x_1 + x_2 = 1$ $\qquad A = [1 \quad 1]$. $\qquad b = [1]$

$x_1 \geq 0, \; x_2 \geq 0$.



$\beta = (1)$ $\qquad A_\beta = [1]$ $\qquad x_\beta : \; A_\beta x_\beta = b$ $\qquad 1 \cdot x_\beta = 1$ $\qquad x_\beta = x_1 = 1$

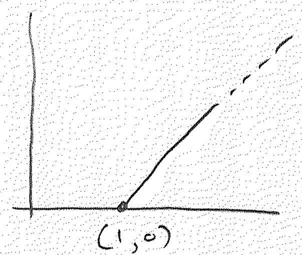
$x_\nu = x_2 = 0$ $\qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\beta = (2)$ $\qquad A_\beta = [1]$ $\qquad x_\beta : \; A_\beta x_\beta = b$ $\qquad 1 \cdot x_\beta = 1$ $\qquad x_\beta = x_2 = 1$

$x_\nu = x_1 = 0$ $\qquad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(2) $x_1 - x_2 = 1$ $\qquad A = [1 \quad -1]$ $\qquad b = [1]$

$x_1 \geq 0, \; x_2 \geq 0$



$\beta = (1)$ $\qquad A_\beta = [1]$ $\qquad x_\beta : \; A_\beta x_\beta = b$ $\qquad 1 \cdot x_1 = 1$ $\qquad x_1 = 1$

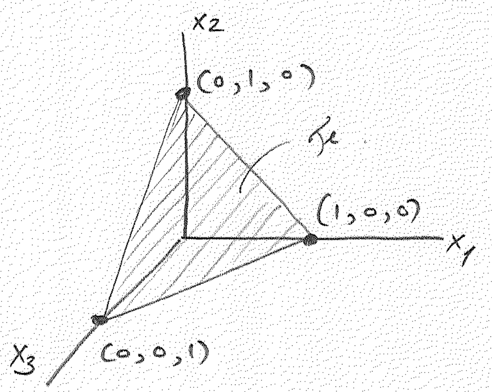
$x_\nu : \; x_2 = 0$ $\qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\beta = (2)$ $\qquad A_\beta = [-1]$ $\qquad x_\beta : \; A_\beta x_\beta = b$ $\qquad -1 \cdot x_2 = 1$ $\qquad x_2 = -1$

$x_\nu : \; x_1 = 0$ $\qquad x = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ basic, but not basic feasible.

(3) $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \end{bmatrix}$

$$\mathcal{X} = \left\{ x \in \mathbb{R}^3 : \; Ax = b, \; x \geq 0 \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \; x_1 + x_2 + x_3 = 1, \; x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0 \right\}$$



3 basic feasible solutions:

$\beta = (1) \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \beta = (2) \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \beta = (3) \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

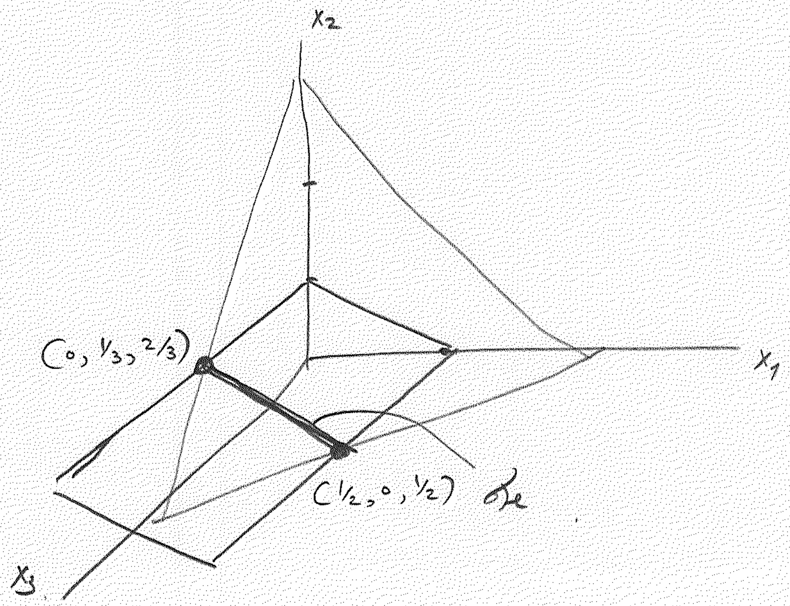
(4) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\mathcal{X} = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \; \begin{array}{l} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 3x_2 = 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{array} \right\}.$$

$$\beta = (1, 2) \quad \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad \underbrace{\beta = (2, 3) \quad \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix} \qquad \beta = (1, 3) \quad \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}}_{\text{basic feasible solutions}}$$

basic, but not basic feasible

basic feasible solutions.



So we expect two corner points of $\mathcal{X}$.

(5) $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$. $\quad \mathcal{X} = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \begin{array}{l} x_1 + x_2 + x_3 = 200 \\ 2x_1 + x_2 + x_4 = 300 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \\ x_4 \geq 0 \end{array} \right\}$

$n = 4$

$m = 2$

$\binom{n}{m} = \binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = \frac{12}{2} = 6.$

$\beta = (1, 2)$	$(1, 3)$	$(1, 4)$	$(2, 3)$	$(2, 4)$	$(3, 4)$
$\begin{bmatrix} 100 \\ 100 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 150 \\ 0 \\ 50 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 200 \\ 0 \\ 0 \\ -100 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 300 \\ -100 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 200 \\ 0 \\ 100 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 200 \\ 300 \end{bmatrix}$

4 basic feasible solutions.

basic, but not feasible

Let us revisit our picture in $\mathbb{R}^2$ for the original problem (before we converted it into standard form)