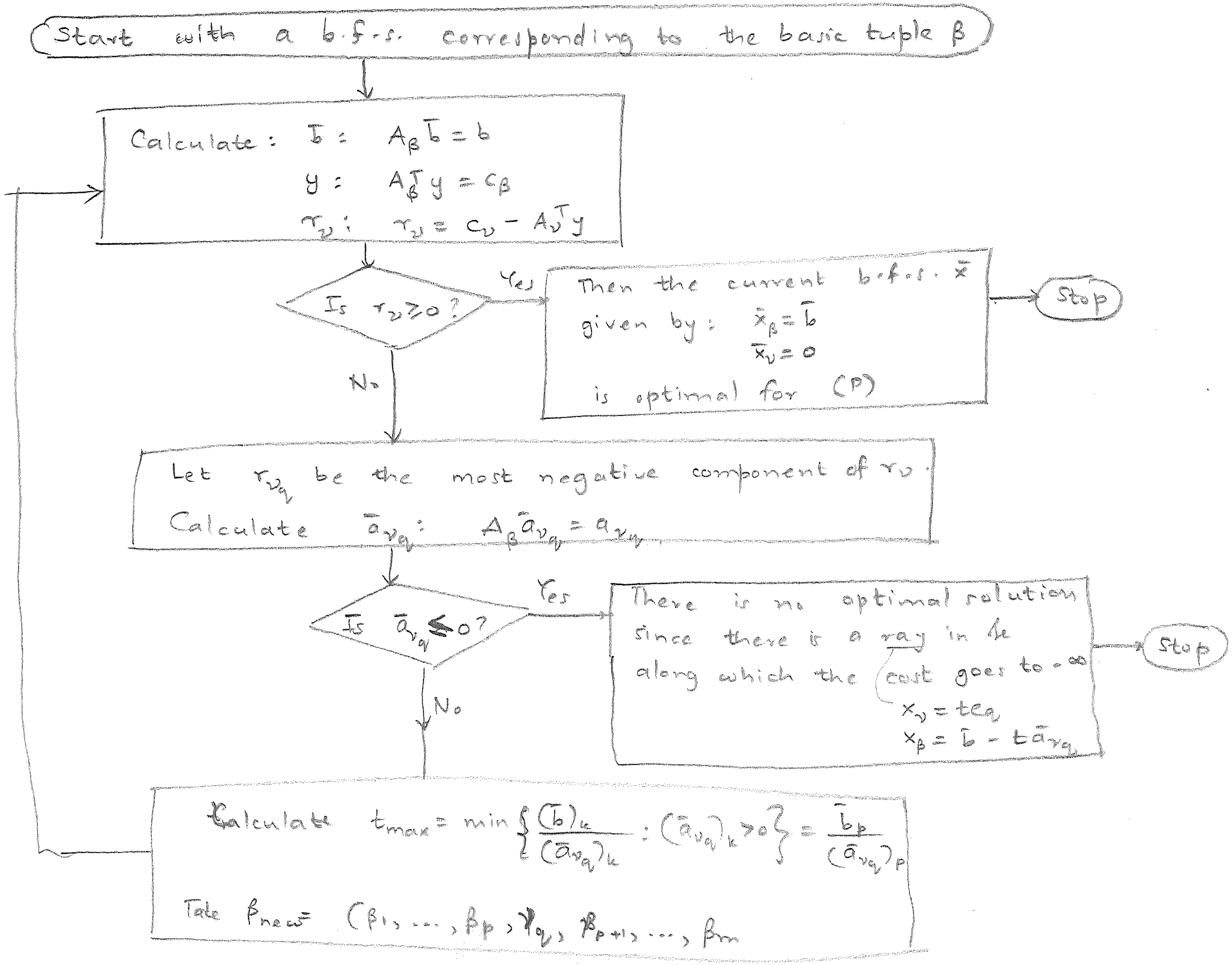
$$(P): \begin{cases} \text{Minimize } c^T x \\ \text{s.t. } \quad Ax = b \\ \qquad\quad x \geq 0 \end{cases}$$

Simplex method



<u>Example:</u>

$$\begin{cases} \text{minimize} & 4x_1 + x_2 - x_3 + 2x_4 \\ \text{s.t.} & 3x_1 - 3x_2 + x_3 = 3 \\ & 6x_1 - 2x_2 + x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

$$c = \begin{bmatrix} 4 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -3 & 1 & 0 \\ 6 & -2 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Start with slack variables as the basic variables.

$\beta = (3, 4)$. $\quad \nu = (1, 2)$

$$A_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A_\nu = \begin{bmatrix} 3 & -3 \\ 6 & -2 \end{bmatrix}.$$

<u>Initial b.f.s:</u>

x_β: $\quad A_\beta x_\beta = b$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \geq 0$$

x_ν: $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix} \geq 0.$$

<u>Is it optimal?</u>

- Calculate y
- Red. costs r_ν.

<u>Simplex multipliers vector y:</u> $A_\beta^T y = c_\beta$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow y = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

<u>Reduced costs of the nonbasic variables r_ν:</u> $r_\nu = c_\nu - A_\nu^T y$.

$$r_\nu = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 9 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}.$$

<u>Is $r_\nu \geq 0$?</u> Answer: no.

So we can't conclude that the current b.f.s. $\bar{x}$ is optimal.

<u>Have we found a ray starting at $\bar{x}$ along which the cost goes to $-\infty$?</u>

Take the most negative component r_{ν_q} of r_ν.

In our case, $q = 1$, $\nu_q = \nu_1 = 1$.

Then calculate $\bar{a}_{\nu_q}$ using $A_\beta \bar{a}_{\nu_q} = a_{\nu_q}$:

So: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{a}_1 = a_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

Thus $\bar{a}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

<u>Ask:</u> Is $\bar{a}_{\nu_q} \leq 0$? <u>Answer:</u> No.

Finding a new, better b.f.s. :

Calculate $t_{max} = \min \left\{ \frac{\bar{b}_k}{\bar{a}_{\nu_q, k}} : \bar{a}_{\nu_q, k} > 0 \right\}$, and find a minimizing index p,

i.e., $\frac{\bar{b}_p}{\bar{a}_{\nu_q, p}} = t_{max}$.

Take $\beta_{new} = (\beta_1, \dots, \beta_{p-1}, \boxed{\nu_q}, \beta_{p+1}, \dots, \beta_m)$ as the new basic tuple.

In our case, $t_{max} = \min \left\{ \frac{3}{3}, \frac{2}{6} \right\} = \min \left\{ 1, \frac{1}{3} \right\} = \frac{1}{3} = \frac{\bar{b}_2}{\bar{a}_{1,2}}$.

So $p = 2$, $\beta_p = \beta_2 = 4$. leaves the set of basic variables.

$\nu_q = \nu_1 = 1$ enters the set of basic variables.

So the new basic tuple is $\beta_{new} = (3, \boxed{1})$.

What is the new b.f.s.?

Calculate $\bar{b}$ using $A_\beta \bar{b} = b$. $A_\beta = \begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix} \bar{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \bar{b} = \begin{bmatrix} 2 \\ 1/3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix}.$$

$$x_\nu = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $x = \begin{bmatrix} 1/3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ is the new b.f.s.

Is the new b.f.s. optimal? Calculate y, r_ν.

$\underline{y}$: $A_B^T y = c_B$. $\begin{bmatrix} 1 & 0 \\ 3 & 6 \end{bmatrix} y = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \Rightarrow y = \begin{bmatrix} -1 \\ 7/6 \end{bmatrix}$

$\underline{r_\nu}$: $r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ -2 & 1 \end{bmatrix}^T \begin{bmatrix} -1 \\ 7/6 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 7/6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 & -7/3 \\ & 7/6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 7/6 \end{bmatrix}$

$= \begin{bmatrix} 1/3 \\ 5/6 \end{bmatrix} \geq 0$.

So the new b.f.s. is optimal,

and the simplex algorithm terminates.

We have found the foll. optimal solution:

$$x = \begin{bmatrix} 1/3 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

Example

$$\begin{cases} \text{minimize} & -x_2 \\ \text{subject to} & x_1 - x_2 \leq 2 \\ & -2x_1 + x_2 \leq 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases} \xrightarrow{\text{convert to standard form}} \begin{cases} \text{minimize} & -x_2 \\ \text{subject to} & x_1 - x_2 + x_3 = 2 \\ & -2x_1 + x_2 + x_4 = 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \\ & x_4 \geq 0 . \end{cases}$$

$$A = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \qquad b = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} .$$

Starting basic feasible solution : $\beta = (3, 4)$ $\quad A_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \bar{b} = I^{-1} b = b = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \geq 0 .$

So feasible.

$\nu = (1, 2)$ $\quad A_\nu = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} .$

First iteration

(1) Find $\bar{b}, y, r_\nu$.

$\bar{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \geq 0 .$

y using $A_\beta^T y = c_\beta$: $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So $y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

r_ν using $r_\nu = c_\nu - A_\nu^T y$: $\quad r_\nu = \begin{bmatrix} 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} .$

(2) $\neg [r_\nu \geq 0]$.

So we can't conclude that the current b.f.s. is optimal.

$r_{\nu_2} = r_2 = -1$ is the most negative component of r_ν.

So $q = 2$ and $\nu_q = \nu_2 = 2$.

We need to calculate $\bar{a}_{\nu_q} = \bar{a}_{\nu_2} = \bar{a}_2$ using $A_\beta \bar{a}_{\nu_q} = a_{\nu_q}$ i.e., $A_\beta \bar{a}_2 = a_2$

So $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{a}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and so $\bar{a}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(3) $\neg [\bar{a}_2 \leq 0]$. So we must find t_{max} and p.

$$t_{max} = \min \left\{ \frac{(\bar{b})_k}{(\bar{a}_{\nu_q})_k} : \bar{a}_{\nu_q,k} > 0 \right\} = \min \left\{ \frac{0}{1} \right\} = \frac{0}{1} = \frac{\bar{b}_2}{\bar{a}_{2,2}}$$

So $p = 2$

Thus $\beta_p = \beta_2 = 4$ leaves the basic tuple, and $\nu_q = \nu_2 = 2$ enters the basic tuple.

So the new basic tuple is $\beta = (3, \boxed{2})$, and $A_\beta = \begin{array}{c} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ \begin{matrix} a_3 & a_2 \end{matrix} \end{array}$.

$4 = \beta_p$

$2 = \nu_q$

Moreover, $\nu = (1, 4)$ and $A_\nu = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \\ \begin{matrix} a_1 & a_4 \end{matrix} \end{array}$.

The First iteration is complete.

Second iteration

(1) Find $\bar{b}$, y, r_ν.

$\underline{\bar{b}}$: $A_\beta \bar{b} = b$ i.e., $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \bar{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. So $\bar{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

$\underline{y}$: $A_\beta^T y = c_\beta$ i.e., $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. So $y = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$\underline{r_\nu}$: $r_\nu = c_\nu - A_\nu^T y = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(2) $\neg [r_\nu \geq 0]$.

$r_{\nu_1} = r_1 = -2$ is the most negative component of r_ν.

So $q = 1$, $\nu_q = \nu_1 = 1$.

We must now calculate $\bar{a}_{\nu_q}$ using $A_\beta \bar{a}_{\nu_q} = a_{\nu_q}$,

i.e., $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \bar{a}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. So $\bar{a}_1 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

Since $\bar{a}_1 \leq 0$, there is no optimal solution

(There is a ray along which the cost goes to $-\infty$.

What is the ray?

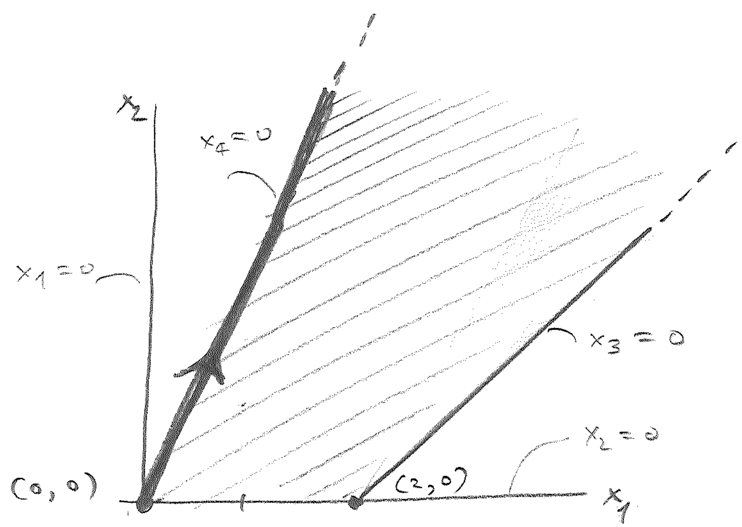
4.8

$t \geq 0$. $\quad x_\nu = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = t e_q = \begin{bmatrix} t \\ 0 \end{bmatrix}$.

$\begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = x_\beta = \bar{b} - t \bar{a}_{\nu_q} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - t \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2+t \\ 2t \end{bmatrix}$.

$x = \begin{bmatrix} t \\ 2t \\ 2+t \\ 0 \end{bmatrix} \geq 0$. $\quad Ax = b$.

Cost of $x = -2t \xrightarrow{t \to +\infty} -\infty$



Duality theory

Associated with every linear programming problem is a "dual" linear programming problem.

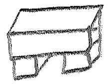
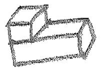
Primal LP problem $\rightsquigarrow$ Dual LP problem

There is a special relationship between the two.

Duality theory

- One obtains important information by looking at the dual problem.
- Sensitivity analysis – how the optimal solution depends on the primal problem parameters.
- Interior point methods use duality.

<u>Example</u> Recall the production planning problem for our furniture company.

Tables		T	400 kr	Big parts		200
Chairs		C	300 kr.	Small parts		300.

<u>Problem:</u>

$$\begin{cases} \text{Maximize profit} & 400T + 300C \\ \text{s.t.} & T + C \leq 200 \\ & 2T + C \leq 300 \\ & T \geq 0 \\ & C \geq 0. \end{cases}$$

Suppose there is another rival furniture company (lets call it IKEA) that wants to buy our resources (big parts and small parts).

IKEA asks itself: What is the least amount it should pay?

Let $x_1 :=$ price at which IKEA buys big parts from us.

$x_2 :=$ price at which IKEA buys small parts from us.

Total price to buy all our resources $= 200 \cdot x_1 + 300 \cdot x_2$.

If we sell one big part and two small parts to IKEA,
we would make $x_1 + 2x_2$,
whereas if we sell a table, we make 400 SEK

So IKEA knows that its x_1, x_2 should be such that

$$x_1 + 2x_2 \geq 400.$$

Similarly $x_1 + x_2 \geq 300$

selling price of chairs.

So IKEA is faced with the following problem:

$$\begin{cases} \text{minimize} & 200x_1 + 300x_2 \\ \text{s.t.} & x_1 + 2x_2 \geq 400 \\ & x_1 + x_2 \geq 300 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{cases}$$

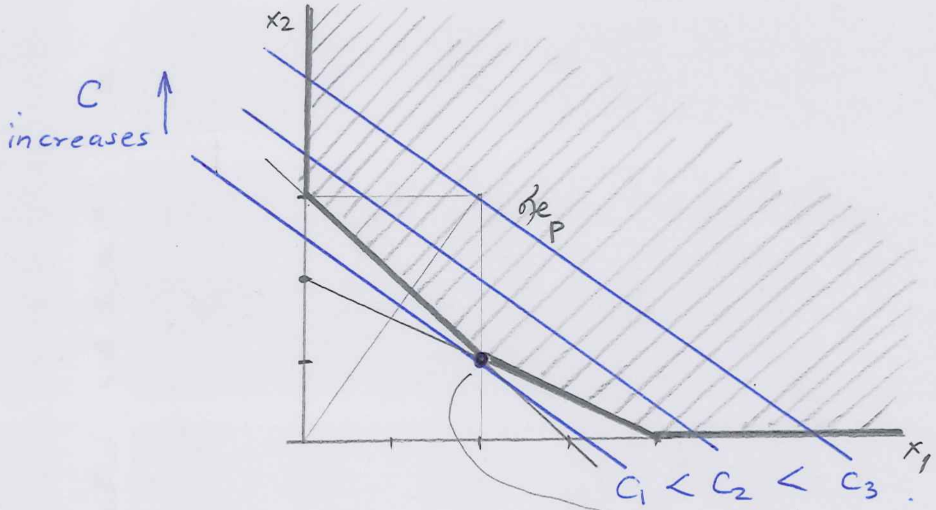
Let us compare this problem with the original problem.

4.12

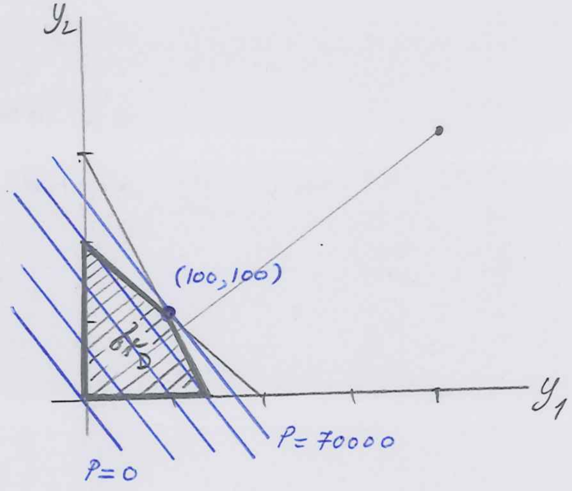
$$\begin{pmatrix} y_1 =: T \\ y_2 =: C \end{pmatrix}$$

$$(P): \begin{cases} \text{Minimize} & 200\,x_1 + 300\,x_2 \\ \text{s.t.} & \boxed{\begin{matrix} x_1 + 2x_2 \\ x_1 + x_2 \end{matrix}} \begin{matrix} \geq \boxed{400} \\ \geq \boxed{300} \end{matrix} \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases}$$

$$(D): \begin{cases} \text{Maximize} & \boxed{400}\,y_1 + \boxed{300}\,y_2 \\ \text{s.t.} & \boxed{\begin{matrix} y_1 + y_2 \\ 2y_1 + y_2 \end{matrix}} \begin{matrix} \leq 200 \\ \leq 300 \end{matrix} \\ & y_1 \geq 0 \\ & y_2 \geq 0. \end{cases}$$



$200\,x_1 + 300\,x_2 = C$



$$\begin{aligned} x_1 + 2x_2 &= 400 \\ x_1 + x_2 &= 300 \\ \hline x_2 &= 100 \\ x_1 &= 200. \end{aligned}$$

Optimal profit $= 400 \cdot 100 + 300 \cdot 100$

$= 40000 + 30000$

$= \boxed{70000 \text{ kr}}$.

Optimal cost $= 200 \cdot 200 + 300 \cdot 100 = \boxed{70000 \text{ kr}}$.

$=$

Consider the LP problem in <u>canonical form</u>:

$$(P): \begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & \boxed{A} x \geq \boxed{b} \\ & x \geq 0 . \end{cases}$$

(Just like the standard form, every LP problem can be rewritten as an LP problem in canonical form).

We define its dual to be the following LP problem:

$$(D): \begin{cases} \text{maximize} & \boxed{b}^T y \\ \text{s.t.} & \boxed{A^T} y \leq c \\ & y \geq 0 . \end{cases}$$

This form is nicer for duality, since the dual has a similar structure as the primal problem.

$$(P): \begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax \geqslant b \\ & x \geqslant 0 \end{cases} \qquad (D): \begin{cases} \text{maximize} & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geqslant 0 \end{cases}$$

$$\mathcal{F}_P = \left\{ x \in \mathbb{R}^n : \begin{array}{l} Ax \geqslant b \\ x \geqslant 0 \end{array} \right\} \qquad \mathcal{F}_D = \left\{ y \in \mathbb{R}^m : \begin{array}{l} A^T y \leq c \\ y \geqslant 0 \end{array} \right\}.$$

Theorem (Weak duality)

If $x \in \mathcal{F}_P$ and $y \in \mathcal{F}_D$, then $c^T x \geqslant b^T y$.

Proof

$$\begin{aligned} c^T x &= (c - A^T y + A^T y)^T x \\ &= (c - A^T y)^T x + y^T A x \\ &= (c - A^T y)^T x + y^T (Ax - b + b) \\ &= \underbrace{(c - A^T y)^T}_{\geqslant 0} \underbrace{x}_{\geqslant 0} + \underbrace{y^T}_{\geqslant 0} \underbrace{(Ax - b)}_{\geqslant 0} + y^T b \\ &\geqslant y^T b = b^T y. \end{aligned}$$

$\square$

Costs of (D) give lower bounds on the optimal cost of (P).

<u>Theorem</u> (Duality theorem)

	$\mathcal{X}_P$	$\mathcal{X}_D$	Conclusion
(1)	$\neq \phi$	$\neq \phi$	$\exists$ an optimal solution $\hat{x}$ to (P) $\exists$ an optimal solution $\hat{y}$ to (D) $c^T\hat{x} = b^T\hat{y}$
(2)	$\neq \phi$	$= \phi$	$\forall\, \rho \in \mathbb{R}$, $\exists\, x \in \mathcal{X}_P$ s.t. $c^T x < \rho$ Neither (P) nor (D) has an optimal solution
(3)	$= \phi$	$\neq \phi$	$\forall\, \rho \in \mathbb{R}$, $\exists\, y \in \mathcal{X}_D$ s.t. $b^T y > \rho$. Neither (P) nor (D) has an optimal solution.
(4)	$= \phi$	$= \phi$	Neither (P) nor (D) has an optimal solution. $\mathbb{R}$