Some linear algebra

Vector space $\Big(\underset{\text{set}}{V}, \underset{\substack{\text{vector}\\ \text{addition}}}{+}, \underset{\substack{\text{scalar}\\ \text{multiplication}}}{\cdot} \Big)$

$$+ : V \times V \longrightarrow V$$
$$(v_1, v_2) \longmapsto v_1 + v_2$$

$$\cdot : \mathbb{R} \times V \longrightarrow V$$
$$(\alpha, v) \longmapsto \alpha \cdot v$$

Vector space axioms:

$+$ $\begin{cases} \text{(V1)} & \forall v_1, v_2, v_3 \in V, \quad v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3 \\ \text{(V2)} & \exists\, 0 \in V \text{ s.t. } \forall v \in V, \quad 0 + v = v = v + 0 \\ \text{(V3)} & \forall v \in V, \ \exists -v \in V \text{ s.t. } v + -v = 0 = -v + v \\ \text{(V4)} & \forall v_1, v_2 \in V, \quad v_1 + v_2 = v_2 + v_1. \end{cases}$

$\cdot$ $\begin{cases} \text{(V5)} & \forall v \in V, \quad 1 \cdot v = v. \\ \text{(V6)} & \forall \alpha, \beta \in \mathbb{R}, \ \forall v \in V, \quad \alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v. \end{cases}$

$+, \cdot$ $\begin{cases} \text{(V7)} & \forall \alpha, \beta \in \mathbb{R}, \ \forall v \in V, \quad (\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v \\ \text{(V8)} & \forall \alpha \in \mathbb{R}, \ \forall v_1, v_2 \in V, \quad \alpha \cdot (v_1 + v_2) = \alpha \cdot v_1 + \alpha \cdot v_2 \end{cases}$

Examples

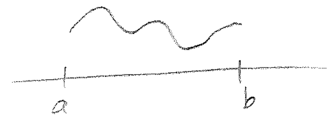
(1) $\mathbb{R}^n$ $\quad V = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \;\middle|\; a_1, \dots, a_n \in \mathbb{R} \right\}$

$+$: $\quad u = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad v = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad u + v := \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}.$

$\cdot$: $\quad \alpha \in \mathbb{R} \quad \alpha . u = \alpha . \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} := \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix}.$

Zero vector? $\quad \underline{0} := \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$

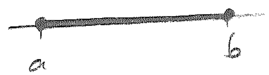
(2) $\mathbb{R}^{[a,b]}$ $\quad V = \{ f : [a,b] \longrightarrow \mathbb{R} \}$



$+$: $\quad f, g \in V \quad f + g \in V$ is defined by $(f+g)(t) := f(t) + g(t) \quad (t \in [a,b])$

$\cdot$: $\quad \alpha \in \mathbb{R} \quad \alpha . f \in V$ is defined by $(\alpha . f)(t) := \alpha f(t) \quad (t \in [a,b]).$

Zero vector? $\quad \underline{0} \in V$, where $\underline{0}(t) = 0 \quad \forall\, t \in [a,b]$.
zero function



Linear independence

Vectors $v_1, \dots, v_n$ in V are called <u>linearly independent</u> if

whenever $\alpha_1, \dots, \alpha_n$ are scalars such that $\alpha_1 \cdot v_1 + \dots + \alpha_n \cdot v_n = \underline{0}$,

then $\alpha_1 = \dots = \alpha_n = 0$.

(So the only linear combination of $v_1, \dots, v_n$ giving rise to the zero vector is when all the scalars are zero)

Vectors $v_1, \dots, v_n$ in V are called <u>linearly dependent</u> if

there exist scalars $\alpha_1, \dots, \alpha_n$, not all zeros, such that $\alpha_1 \cdot v_1 + \dots + \alpha_n \cdot v_n = \underline{0}$.

<u>Examples</u>.

(1) The vectors $e_1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ are linearly independent in $\mathbb{R}^n$

since if there are scalars $\alpha_1, \dots, \alpha_n$ s.t. $\alpha_1 \cdot e_1 + \dots + \alpha_n \cdot e_n = 0$

i.e., $\begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, then $\alpha_1 = \dots = \alpha_n = 0$.

The vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, v_{n-1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$ are linearly independent in $\mathbb{R}^n$.

The vectors $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ are linearly dependent in $\mathbb{R}^3$, since

$$1 \cdot \begin{bmatrix}0\\0\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = \underline{0}.$$

(2) The vectors $1, t, t^2$ are linearly independent in $\mathbb{R}^{[0,1]}$.

If $\quad \alpha \cdot 1 + \beta \cdot t + \gamma \cdot t^2 = \underline{0} \quad (*)$

then:

$t = 0$: $\quad \alpha = 0$.

Diff. $(*)$: $\quad \beta + 2\gamma t = 0$

$t = 0$: $\quad \beta = 0$.

Diff. twice $(*)$: $\quad 2\gamma = 0$.

So $\gamma = 0$.

The vectors $1, t, 4 - 7t$ are linearly dependent in $\mathbb{R}^{[0,1]}$,

$$-4 \cdot 1 + (-7) \cdot t + 1 \cdot (4 - 7t) = 0 \qquad (t \in [0,1])$$

Subspace of a vector space $S \subset V$ is called a **subspace** of V if

S1. $0 \in V$

S2 $\forall\, v_1, v_2 \in S,\ v_1 + v_2 \in S.$

S3. $\forall\, v \in S\ \ \forall\, \alpha \in \mathbb{R},\ \ \alpha . v \in S.$

Examples

(1) If $v_1, \dots, v_k \in V$, then

$$\operatorname{span}\{v_1, \dots, v_k\} := \{\alpha_1 v_1 + \dots + \alpha_k v_k : \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$$

is a subspace of V.

(2) The set of polynomials of degree at most 2010

$$\mathcal{P} = \operatorname{span}\{1, t, t^2, \dots, t^{2010}\}$$

is a subspace of $\mathbb{R}^{[0,1]}$.

Basis

A <u>basis</u> of a vector space V is a set $B \subset V$ such that

(1) $\operatorname{span} B = V$

(2) B is linearly independent.

Examples

(1) A basis for $\mathbb{R}^n$ is $B = \{e_1, \dots, e_n\}$, where $e_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← kth position, $k = 1, \dots, n$.

i.e., $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$.

Another basis is $\tilde{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$.

(2) A basis for the vector space of all polynomials of degree ≤ 2010 is $\{1, t, t^2, \dots, t^{2010}\}$.

(3) A basis for the subspace $S = \left\{ x = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n : a_1 + \dots + a_n = 0 \right\}$ of $\mathbb{R}^n$ is the set $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

Although a nonzero vector space has infinitely many bases, the number of elements in each basis is the same. This number is called the <u>dimension</u> of the vector space.

<u>Examples</u> (1) $\dim . \mathbb{R}^n = n$.

(2) dim. of the vector space of all polynomials of degree ≤ 2010 $= 2011$.

(3) $\dim . \left\{ x = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n : a_1 + \dots + a_n = 0 \right\} = n - 1.$

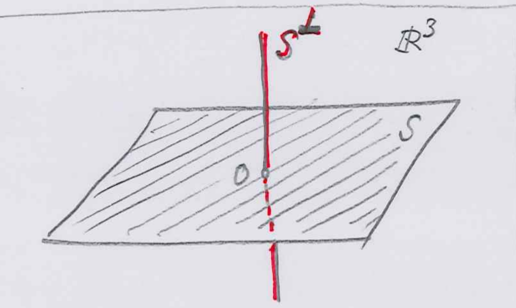
<u>Orthogonal complement of a subspace of $\mathbb{R}^n$</u>

$S \subset \mathbb{R}^n$; S is a subspace of $\mathbb{R}^n$.

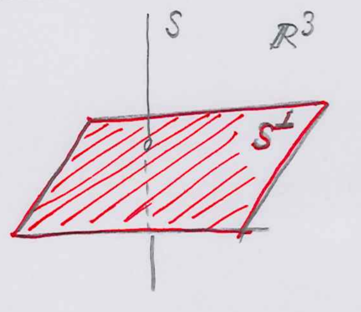
Define the <u>orthogonal complement</u> $S^\perp$ of S as

$$S^\perp = \{ \xi \in \mathbb{R}^n : \xi^T x = 0 \quad \forall x \in X \}$$

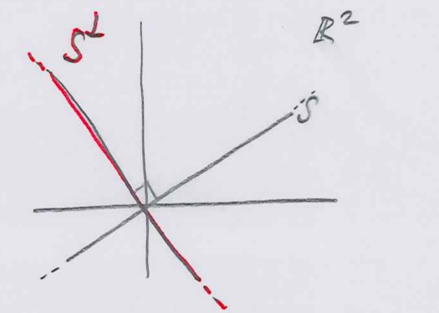
<u>Examples:</u> (1)



(2)



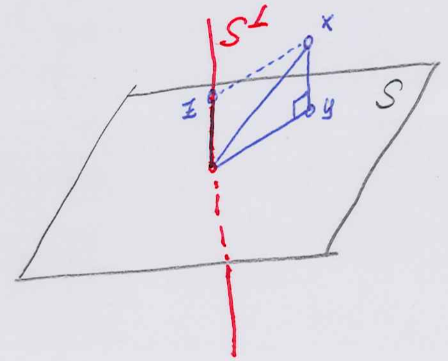
(3)



<u>Theorem</u>. Let S be a subspace of $\mathbb{R}^n$. Then:

(1) $\forall x \in \mathbb{R}^n$, $\exists!\ y \in S$ and $\exists!\ z \in S^\perp$ s.t. $x = y + z$.

(∃! means "there exists a unique")



(2) $(S^\perp)^\perp = S$.

(3) If $\dim. S = k$, then $\dim S^\perp = n - k$.

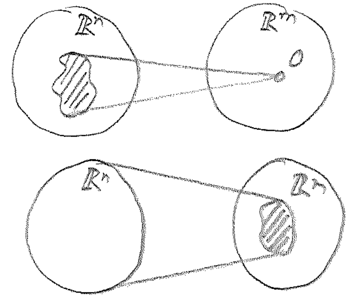
$A \in \mathbb{R}^{m\times n}$.

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (Linear transformation: $A(v_1+v_2) = Av_1 + Av_2$, $A(\alpha v) = \alpha Av$.)

$x \longmapsto Ax$.

$\ker A := \{x \in \mathbb{R}^n : Ax = 0\}$ subspace of $\mathbb{R}^n$.

$\operatorname{ran} A := \{Ax : x \in \mathbb{R}^n\}$



<u>Theorem</u>. Let $A \in \mathbb{R}^{m\times n}$. Then $(\operatorname{ran} A)^\perp = \ker A^T$.

<u>Proof</u>. <u>$(\operatorname{ran} A)^\perp \subset \ker A^T$:</u>

$\eta \in (\operatorname{ran} A)^\perp$.

$\Rightarrow \forall y \in \operatorname{ran} A, \quad \eta^T y = 0$.

$\Rightarrow \forall x \in \mathbb{R}^n, \quad \eta^T A x = 0$.

Let $x = A^T\eta$.

Then $\eta^T A \underbrace{A^T\eta} = 0$

$[\xi_1, \dots \xi_n]\begin{bmatrix}\xi_1\\ \vdots \\ \xi_n\end{bmatrix} = 0$ i.e, $\xi_1^2 + \dots + \xi_n^2 = 0$

So $\xi_1 = \dots = \xi_n = 0$,

i.e., $A^T\eta = 0$

i.e., $\eta \in \ker A^T$.

<u>$\ker A^T \subset (\operatorname{ran} A)^\perp$:</u>

$\eta \in \ker A^T$.

$\Rightarrow A^T\eta = 0$.

Let $y \in \operatorname{ran} A$. (We want $\eta^T y = 0$)

Then $\exists x$ s.t. $y = A.x$.

So $\eta^T y = \eta^T A x = (A^T\eta)^T x = 0^T x = 0$.

So $\eta \in (\operatorname{ran} A)^\perp$.

$\square$

$A \in \mathbb{R}^{m \times n}$

How do we calculate a basis for $\operatorname{ran} A$ and $\ker A$?

(We will need this in Quadratic Optimization.)

<u>Theorem</u>. Let $A \in \mathbb{R}^{m \times n}$ and

let $A = BC$ where $B \in \mathbb{R}^{m \times r}$ has independent columns,

$C \in \mathbb{R}^{r \times n}$ has independent rows.

Then: (1) $\ker A = \ker C$

(2) $\operatorname{ran} A = \operatorname{ran} B$

(3) A has rank r.

<u>Proof</u>. (1) $x \in \ker A \Rightarrow Ax = 0 \Rightarrow B(Cx) = 0 \overset{(\because\, B \text{ has indep. columns})}{\Rightarrow} Cx = 0 \Rightarrow x \in \ker C$.

$x \in \ker C \Rightarrow Cx = 0 \Rightarrow BCx = 0 \Rightarrow Ax = 0 \Rightarrow x \in \ker C$.

(2) $A^T = \underset{\text{indep. col}}{C^T}\,\underset{\text{indep. rows}}{B^T} \Rightarrow \ker A^T = \ker B^T \Rightarrow \underset{\displaystyle \operatorname{ran} A}{\underset{\parallel}{(\ker A^T)^\perp}} = \underset{\displaystyle \operatorname{ran} B}{\underset{\parallel}{(\ker B^T)^\perp}}$.

(3) The columns of B span $\operatorname{ran} B$. ($\because\; Bx = \sum_{i=1}^{r} x_i b_i$ where $B = [b_1 \cdots b_r]$.)

Also they are independent. So they form a basis for $\operatorname{ran} B$.

So $\dim(\operatorname{ran} B) = r$. But $\operatorname{ran} A = \operatorname{ran} B$. So $\dim(\operatorname{ran} A) = r$, i.e., the rank of A is r. $\square$.

The converse of this result is also true:

If A has rank r, then $A = BC$, where $B \in \mathbb{R}^{m\times r}$ has independent columns, and $C \in \mathbb{R}^{r \times n}$ has independent rows.

We will show this constructively, using the Gauss-Jordan method.

Recall: By an elementary row transformation we mean an operation on A of one of the following types:

(1) add $\alpha \cdot (\text{row } j)$ to $(\text{row } i)$. (Corresponds to premultiplying A by

$$E := \begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{bmatrix} 1 & & \\ & \ddots \;\; \alpha & \\ & & 1 \end{bmatrix} \end{array}.)$$

(2) interchange row i and row j. (Corresponds to premult. A by

$$E = \begin{array}{cc} & \begin{array}{cccccc} & & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots \;\; 1 \end{bmatrix} \end{array}.)$$

(3) multiply row i by a nonzero scalar c.

(Corresponds to premult. A by

$$E = \begin{array}{cc} & \begin{array}{ccccc} & & i & & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \\ \end{array} & \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array}.)$$

Note that in each of the above three instances, E is invertible.

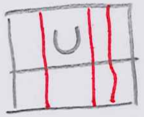
So with some invertible P, we have:

$$PA = \begin{array}{c} \begin{matrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_r \end{matrix} \\ \left[\begin{array}{c} \begin{matrix} 1 * \cdots * & 0 * \cdots * & 0 * & 0 * * & 0 * & 0 & 0 * & \cdots * \\ & 1 * \cdots * & 0 & 0 & 0 & 0 & 0 & \vdots \\ & & 1 * & 0 & 0 & 0 & 0 & \vdots \\ & & & 1 * * & 0 & 0 & 0 & \vdots \\ & & & & 1 * & 0 & 0 & \vdots \\ & & & & & 1 & 0 & \vdots \\ & & & & & & 1 * & \cdots * \end{matrix} \\ \hline O \end{array}\right] \end{array} = \left[\begin{array}{c} U \\ \hline O \end{array}\right].$$

We will take $B = U$ in our desired factorization $A = BC$.

U has independent rows, since if $y^T U = 0$, then $[0 \cdots 0 \; y_1 * \cdots * \; y_2 * \cdots * \; y_r *] = 0 \Rightarrow y = 0$.

We will take $C = [\underbrace{a_{\beta_1} \cdots a_{\beta_r}}_{A_\beta}]$.

C has independent columns: $PA =$ 

$$PA_\beta = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

If $A_\beta v = 0$, then $PA_\beta v = 0$ i.e., $\left[\begin{array}{c} I \\ \hline 0 \end{array}\right] v = 0$. So $v = 0$.

Is $A = BC$? $A = P^{-1}\left[\begin{array}{c} U \\ \hline 0 \end{array}\right] = [S_1 \mid S_2]\left[\begin{array}{c} U \\ \hline 0 \end{array}\right] = S_1 U$.

$A_\beta = [S_1 \mid S_2]\left[\begin{array}{c} I \\ \hline 0 \end{array}\right] = S_1$.

$\Rightarrow A = A_\beta U$.

has indep. columns — has indep. rows.

So (1) $\ker A = \ker U$

(2) $\operatorname{ran} A = \operatorname{ran} A_\beta$.

(3) $\operatorname{rank} A = r$.

$$\left.\begin{array}{l}\text{Columns of } A_\beta \text{ span } \operatorname{ran} A_\beta. \\ \text{Columns of } A_\beta \text{ are independent}\end{array}\right\} \Rightarrow \text{they form a basis for } \operatorname{ran} A_\beta = \operatorname{ran} A.$$

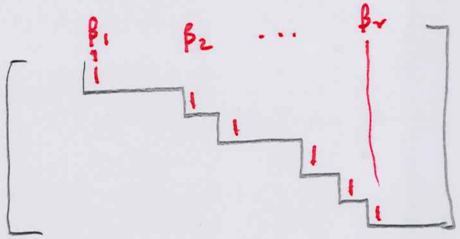
So $\boxed{\text{a basis for } \operatorname{ran} A \text{ is } \{a_{\beta_1}, \dots, a_{\beta_r}\}.}$

<u>ker A ?</u>

$$\ker A = \ker U = \{x \in \mathbb{R}^n : Ux = 0\}$$

$$= \{x \in \mathbb{R}^n : x_\beta + U_\nu x_\nu = 0\}$$

$$= \{x \in \mathbb{R}^n : x_\beta = -U_\nu x_\nu,\ x_\nu \in \mathbb{R}^{n-r}\}.$$

$U =$ 

U_ν = left over columns of U after the β_1st, ..., β_rth columns are deleted.

Set $x_\nu = e_1, \dots, e_{n-r} \in \mathbb{R}^{n-r}$ successively.

Find corresponding x's, call them $z_1, \dots, z_{n-r}$.

<u>Claim</u> : $z_1, \dots, z_{n-r}$ are independent.

Indeed if $\alpha_1 z_1 + \dots + \alpha_{n-r} z_{n-r} = 0$

then by looking at the ν_1st, ..., ν_{n-r}th components in the above equation,

we see that $\alpha_1 = 0, \dots, \alpha_{n-r} = 0$.

But by the rank - nullity theorem (for $A \in \mathbb{R}^{m \times n}$, $\dim(\ker A) + \operatorname{rank} A = n$)

we have $\dim(\ker A) = n - r$.

Since $z_1, \dots, z_{n-r}$ are independent vectors in $\ker A$

it follows that they form a basis for $\ker A$.

So $\boxed{\text{a basis for } \ker A \text{ is } \{z_1, \dots, z_{n-r}\}}$.

<u>Example</u>.

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 3 & 3 \\ 3 & 6 & 4 & 5 \end{bmatrix}$$

(Add $(-2)\cdot$(row 1) to (row 2);
Add $(-3)\cdot$(row 1) to (row 3))

$$E_1 A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

(Multiply row 2 by -1)

$$E_2 A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

(Add $(-2)\cdot$(row 2) to (row 1);
Add $2\cdot$(row 2) to (row 3))

$$PA = \begin{array}{cccc} 1 & & 3 & \\ \end{array}$$
$$PA = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$ has independent rows.

$$\left(0 = y^T U = [y_1 \;\; y_2] \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} = [y_1 \;\; * \;\; y_2 \;\; *] \Rightarrow y_1 = y_2 = 0 \Rightarrow y = 0. \right)$$

$$A_\beta = [a_1 \;\; a_3] = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

(A_β has independent columns:

$$PA = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 0 \end{bmatrix}$$

$$PA_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 \\ \hline 0 \; 0 \end{bmatrix}.$$

If $A_\beta v = 0$, then $PA_\beta v = 0$ i.e., $\begin{bmatrix} I_2 \\ \hline 0 \; 0 \end{bmatrix} v = 0$

i.e., $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

i.e., $\begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $v_1 = v_2 = 0$.

So $v = 0$.)

<u>That $A = A_\beta U$:</u>

$$A = P^{-1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 0 \end{bmatrix} = \left[S_1 \middle| S_2 \right] \begin{bmatrix} U \\ \hline 0 \; 0 \; 0 \; 0 \end{bmatrix} = S_1 U.$$

$$A_\beta = P^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \left[S_1 \middle| S_2 \right] \begin{bmatrix} I_2 \\ \hline 0 \end{bmatrix} = S_1.$$

So $A = A_\beta S_1$.

Thus (1) $\ker A = \ker U$

(2) $\operatorname{ran} A = \operatorname{ran} A_\beta$

(3) $\operatorname{rank} A = r = 2$.

Columns of A_β span $\operatorname{ran} A_\beta$.
Columns of A_β are independent $\Big\} \Rightarrow$ columns of A_β form a basis for $\operatorname{ran} A_\beta = \operatorname{ran} A$.

So $\boxed{\text{a basis for } \operatorname{ran} A \text{ is } \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix} \right\}}$.

<u>ker A?</u>

$$\ker A = \ker U = \left\{ x \in \mathbb{R}^4 : Ux = 0 \right\} = \left\{ x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} : \begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \right\}.$$

$$= \left\{ x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} : \begin{bmatrix}x_1\\x_3\end{bmatrix} + \underbrace{\begin{bmatrix}2 & 3\\ 0 & -1\end{bmatrix}}_{U_\nu}\begin{bmatrix}x_2\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \right\}$$

$$= \left\{ x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} : \begin{bmatrix}x_1\\x_3\end{bmatrix} = -\underbrace{\begin{bmatrix}2 & 3\\ 0 & -1\end{bmatrix}}_{U_\nu}\begin{bmatrix}x_2\\x_4\end{bmatrix} \right\} \quad x_2, x_4 \in \mathbb{R}$$

We take $x_\nu = \begin{bmatrix}x_2\\x_4\end{bmatrix}$ successively as $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$.

Calculate corresponding x_β:

$$x_\beta = \begin{bmatrix}x_1\\x_3\end{bmatrix} = -\begin{bmatrix}2 & 3\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix}$$

$$x_\beta = \begin{bmatrix}x_1\\x_3\end{bmatrix} = -\begin{bmatrix}2 & 3\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-3\\1\end{bmatrix}.$$

Put x_β, x_ν together to form $z_1, \dots, z_{n-r}$.

$$z_1 = \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \quad z_2 = \begin{bmatrix}-3\\0\\1\\1\end{bmatrix}.$$

z_1, z_2 are linearly independent: If $\alpha_1 z_1 + \alpha_2 \cdot z_2 =$

$$\begin{bmatrix} \alpha_1(-2) + \alpha_2(-3) \\ \alpha_1 \\ \alpha_1(0) + \alpha_2(1) \\ \alpha_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \leftarrow \nu_1 = 2 \\ \\ \leftarrow \nu_2 = 4 \end{array}$$

then if we look at the ν_1st and ν_2nd component, i.e. the 2nd and 4th component, we get $\alpha_1 = \alpha_2 = 0$.

$\dim(\ker A) = n - r = 4 - 2 = 2$.

z_1, z_2 are independent vectors in $\ker A$. $\Big\} \Rightarrow \{z_1, z_2\}$ forms a basis for $\ker A$.

So $\boxed{\text{a basis for } \ker A \text{ is given by } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.}$