1.(a). The problem can be rephrased as

$$\begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & h(x)=0 \end{cases}$$

where $f(x) := (x_1-2)^2 + x_2^2$

$h(x) := x_1^2 + x_2^2 - 1$

We have

$$\nabla h(x) = [2x_1 \quad 2x_2].$$

$\nabla h(x)$ is independent iff $\nabla h(x) \neq 0$.

If $x \in \mathcal{F} := \{x \in \mathbb{R}^2 : h(x)=0\}$, then $(x_1, x_2) \neq (0,0)$. So $\nabla h(x) = [2x_1 \quad 2x_2] \neq [0 \quad 0]$.

Hence every feasible x is a regular point.

Hence if x is an optimal solution, then there exists a $u \in \mathbb{R}$ s.t. $\nabla f(x) + u \nabla h(x) = 0$. (*)

We have $\nabla f(x) = [2(x_1-2) \quad 2x_2]$

So (*) becomes

$$[2(x_1-2) \quad 2x_2] + u[2x_1 \quad 2x_2] = [0 \quad 0].$$

Hence we obtain

$$2(x_1-2) + u \cdot 2x_1 = 0$$
$$2x_2 + u \cdot 2x_2 = 0.$$

So

$$(1+u)x_1 = 2 \qquad (**)$$
$$(1+u)x_2 = 0. \qquad (***)$$

From (**), we see that $1+u \neq 0$. So (***) implies that $x_2 = 0$. Finally from $x_1^2 + x_2^2 = 1$, we obtain that $x_1 = +1$ or -1.

So possible optimal solutions are $(1,0)$ and $(-1,0)$.

Also, $f(1,0) = 1 < f(-1,0) = 9$. So the only possibility for an optimal solution is $(1,0)$.

The feasible set $\mathcal{F}$, is bounded (indeed, $\mathcal{F}$ is contained in the ball with center 0 and radius 1), and it is also closed. So $\mathcal{F}$ is compact. The map $x \xrightarrow{f} (x_1-2)^2 + x_2^2$ is continuous. So we know that $f : \mathcal{F} \to \mathbb{R}$ has a global minimizer on $\mathcal{F}$, by the Weierstrass Theorem.

Hence $(1,0)$ is the unique optimal solution.

1.(b) The problem (P) is in canonical form:

$$(P): \begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{cases}$$

where

$$c = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 2009 \\ 2010 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & & & \ddots & \\ 1 & 1 & \cdots & \cdots & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 2009 \\ 2010 \end{bmatrix}.$$

We have that

$$\hat{x} := 2010\, e_1 = \begin{bmatrix} 2010 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \geq 0$$

and

$$A\hat{x} = \begin{bmatrix} 2010 \\ 2010 \\ 2010 \\ \vdots \\ 2010 \\ 2010 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 2009 \\ 2010 \end{bmatrix} = b,$$

and so $\hat{x}$ is feasible for (P).

The dual problem (D) to (P) is given by:

$$(D): \begin{cases} \text{maximize} & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geqslant 0 \end{cases}$$

i.e.,

$$(D): \begin{cases} \text{maximize} & y_1 + 2y_2 + 3y_3 + \dots + 2010\, y_{2010} \\ \text{s.t.} & y_1 + y_2 + y_3 + \dots + y_{2010} \leq 1 \\ & y_2 + y_3 + \dots + y_{2010} \leq 2 \\ & y_3 + \dots + y_{2010} \leq 3 \\ & \vdots \\ & y_{2009} + y_{2010} \leq 2009 \\ & y_{2010} \leq 2010, \\ & y_1 \geqslant 0,\ y_2 \geqslant 0,\ y_3 \geqslant 0, \dots, y_{2010} \geqslant 0. \end{cases}$$

We have that

$$\hat{y} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \geqslant \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$A^T \hat{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \leqslant \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 2010 \end{bmatrix} = c$$

and so $\hat{y}$ is feasible for (D).

We have $c^T \hat{x} = 2010 + 2 \cdot 0 + 3 \cdot 0 + \dots + 2010 \cdot 0 = 2010$

and $b^T \hat{y} = 0 + 2 \cdot 0 + 3 \cdot 0 + \dots + 2010 \cdot 1 = 2010$.

Since (1) $\hat{x}$ is feasible for (P)

(2) $\hat{y}$ is feasible for (D), and

(3) $c^T \hat{x} = b^T \hat{y}$,

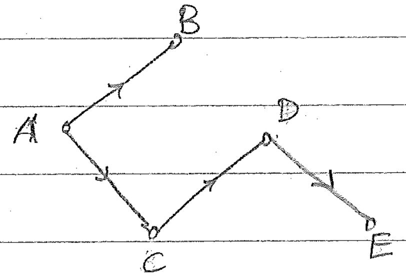
it follows that $\hat{x}$ is optimal for (P).
(By weak duality, for any feasible x for (P), $c^T x \geqslant b^T \hat{y} = c^T \hat{x}$.)

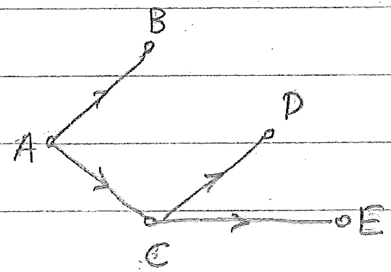
(2) (a) Let m be the number of nodes in the network. Spanning trees are in one-to-one correspondence with a choice of $(m-1)$ independent columns of the incidence matrix of the network. So the number of spanning trees is at most $\binom{n}{m-1}$, where $n :=$ number of edges in the network.

In our case $n = 7$ and $m = 5$, and so the number of spanning trees is at most

$$\binom{7}{5-1} = \binom{7}{4} = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 35.$$

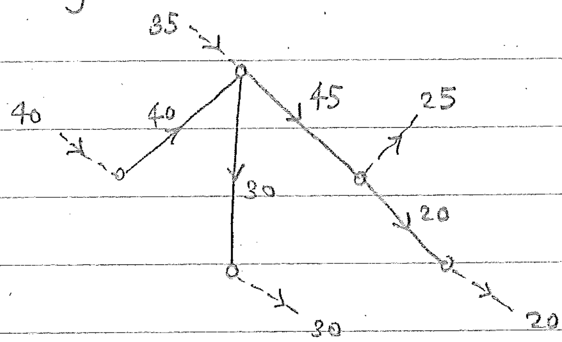
Examples of spanning trees:





The network is balanced since $40+35 = 75 = 30+25+20$.

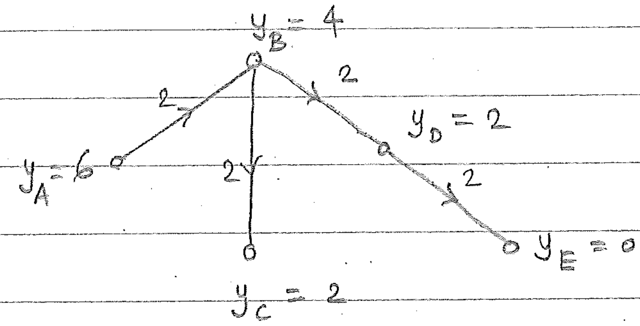
(2).(b). The initial basic solution can be obtained by using flow balance:



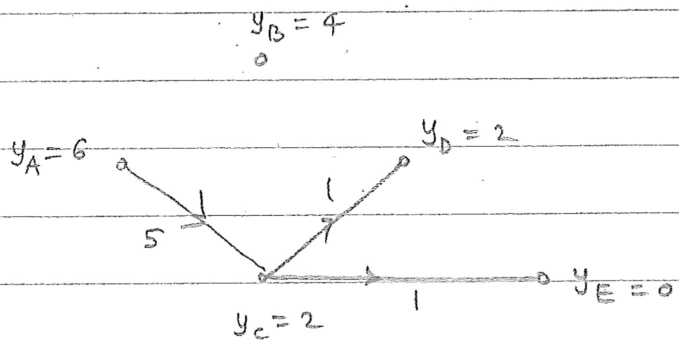
As the flow in each edge is ≥ 0, this solution is also feasible.

We now find the simplex multipliers using

$$\left.\begin{array}{l} c_{ij} = y_i - y_j \\ y_m = 0 \end{array}\right\} \text{ for tree edges } (i,j):$$

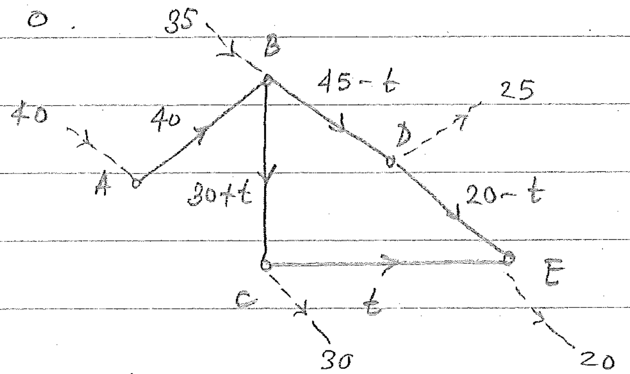


The reduced costs for the nonbasic variables can be found out using $r_{ij} = c_{ij} - (y_i - y_j)$ for nontree edges (i,j):



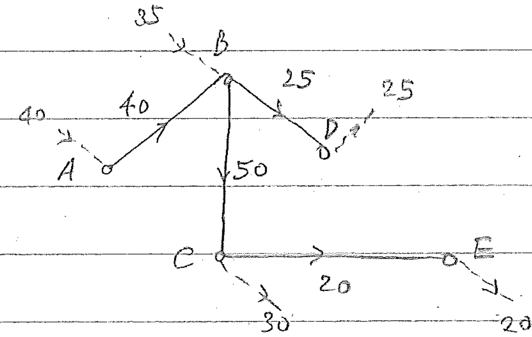
So $r_{AC} = 5 - (6-2) = 5 - 4 = 1 \geq 0$,
$r_{CD} = 1 - (2-2) = 1 - 0 = 1 \geq 0$, and
$r_{CE} = 1 - (2-0) = -1 < 0$.

As $r_{CE} = -1 < 0$, we let $x_{CE} = t$ and let t increase from 0.

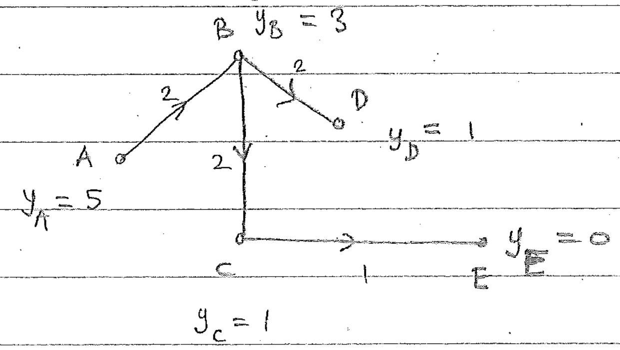


So t can increase to a maximum of 20.

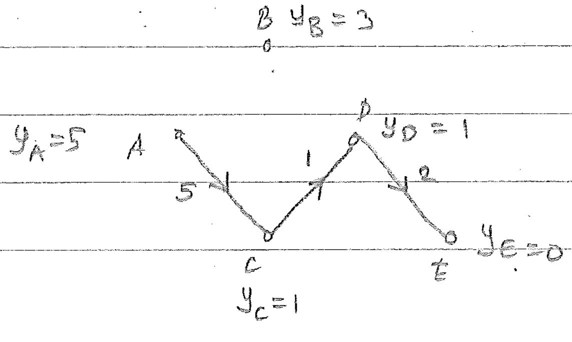
The new basic feasible solution is:



The simplex multipliers vector y can be determined using $\left.\begin{array}{l} c_{ij} = y_i - y_j \\ y_m = 0 \end{array}\right\}$ for tree edges (i,j):



The reduced costs of the nonbasic variables can be found using $r_{ij} = c_{ij} - (y_i - y_j)$ for nontree edges (i,j):



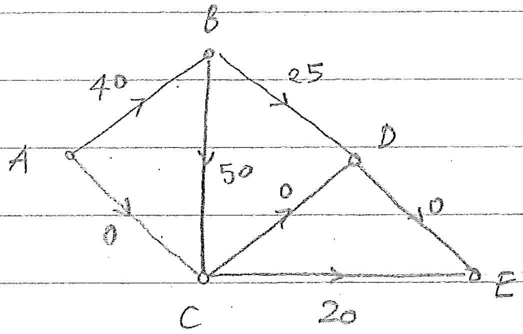
$r_{AC} = 5 - (5 - 1) = 5 - 4 = 1 \geq 0$

$r_{CD} = 1 - (1 - 1) = 1 - 0 = 1 \geq 0$

$r_{DE} = 2 - (1 - 0) = 2 - 1 = 1 \geq 0$

As $r_{ij} \geq 0$, the current basic feasible solution is optimal.

The optimal solution is given by:



The optimal cost is

$$40 \cdot 2 + 50 \cdot 2 + 25 \cdot 2 + 20 \cdot 1$$

$$= 80 + 100 + 50 + 20$$

$$= 250.$$

3.(a). For some suitable invertible E_1, we have

$$E_1 H E_1^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2-\frac{1}{2} & -1-\frac{1}{2} \\ 0 & -1-\frac{1}{2} & 2-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix}.$$

With a suitable invertible E_2, we have

$$E_2 E_1 H E_1^T E_2^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As $2 \geq 0$, $3/2 \geq 0$ and $0 \geq 0$, it follows that H is positive semidefinite.

(Alternately, for $x \in \mathbb{R}^3$,

$$x^T H x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1$$
$$= (x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2 \geq 0.)$$

(b) We perform elementary row transformations to bring A to a "staircase" form:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{\text{add } -3\cdot\text{row1 to row2}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \end{bmatrix}$$

$$\xrightarrow{\text{multiply row 2 by } -1/4} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}.$$

$$\xrightarrow{\text{add } -2\cdot\text{row2 to row1}} \left.\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}\right\} U.$$

Then $\ker A = \ker U = \{x \in \mathbb{R}^3 : Ux = 0\}$

$$= \{x \in \mathbb{R}^3 : x_\beta = -U_\nu x_\nu\}$$

$$= \left\{x \in \mathbb{R}^3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} -1 \\ 2 \end{bmatrix} x_3 \right\}$$

With $x_3 = e_1 = 1$, we have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and so $z_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

So a basis for ker A is given by $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

3.(c) We have

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1 = x^T \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix} x$$

and so the problem can be rewritten as

$$\begin{cases} \text{minimize} & \frac{1}{2} x^T H x + c^T x + c_0 \\ \text{s.t.} & Ax = b \end{cases}$$

where $H := \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $c_0 = 0$.

Let $Z = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

As H is positive semidefinite, so is $Z^T H Z$.

From $Ax = b$, i.e., $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

we see that $\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ clearly satisfies $Ax = b$.

We have

$$Z^T H Z = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} = 3 + 12 + 3 = 18,$$ and

$$Z^T(H\bar{x}+c) = [1 \quad -2 \quad 1](H.0+0) = 0.$$

So the unique solution $\hat{v}$ to $(Z^THZ)\hat{v} = -Z^T(H\bar{x}+c)$ is $\hat{v}=0$. Thus $\hat{x} := \bar{x}+Z\hat{v} = 0+Z0 = 0$ is an optimal solution to the quadratic optimization problem.

3. (d). We have

$$f'(x) = \frac{1}{1+e^{2x}} \cdot 2e^{2x} - 1, \quad \text{and}$$

$$f''(x) = \frac{-1}{(1+e^{2x})^2} \cdot 2e^{2x} \cdot 2e^{2x} + \frac{1}{1+e^{2x}} \cdot 2 \cdot 2e^{2x}$$

$$= \frac{4}{(1+e^{2x})^2}\left(-(e^{2x})^2 + e^{2x}\cdot(1+e^{2x})\right)$$

$$= \frac{4}{(1+e^{2x})^2} \cdot e^{2x}.$$

As $e^r > 0 \ \forall r \in \mathbb{R}$, $f''(x) = \dfrac{4e^{2x}}{(1+e^{2x})^2} > 0 \quad \forall x \in \mathbb{R}$.

We have $f'(x) = \dfrac{2e^{2x}}{1+e^{2x}} - 1 = \dfrac{2e^{2x}-1-e^{2x}}{1+e^{2x}} = \dfrac{e^{2x}-1}{e^{2x}+1}$.

Thus Newton's method gives

$$F(x^{(k)})\left(x^{(k+1)}-x^{(k)}\right) = -\left(\nabla f(x^{(k)})\right)^T$$

i.e., $$\frac{4}{\left(1+e^{2x^{(k)}}\right)^2} \cdot e^{2x^{(k)}}\left(x^{(k+1)}-x^{(k)}\right) = -\frac{e^{2x^{(k)}}-1}{e^{2x^{(k)}}+1}.$$

i.e., $$x^{(k+1)}-x^{(k)} = -\frac{(e^{2x^{(k)}}-1)(e^{2x^{(k)}}+1)}{4e^{2x^{(k)}}}$$

$$= -\frac{(e^{4x^{(k)}}-1)}{4e^{2x^{(k)}}} = -\frac{(e^{2x^{(k)}}-e^{-2x^{(k)}})}{2.2}$$

$$= -\frac{1}{2}\sinh(2x^{(k)})$$

Hence $x^{(k+1)} = x^{(k)} - \dfrac{1}{2}\sinh(2x^{(k)})$.

If $x^{(k)} \xrightarrow{k\to\infty} L$, then $L = L - \frac{1}{2}\sinh(2L)$, i.e., $\sinh(2L) = 0$, i.e., $\frac{e^{2L} - e^{-2L}}{2} = 0$, and so $e^{4L} = 1$. Thus $4L = 0$ and so $L = 0$.

4.(a) Let $p(t) = (t, t, t)$, $t \in \mathbb{R}$.

Then $f(p(t)) = t^2 + t^2 + t^2 - 2 \cdot t \cdot t \cdot t$

$= 3t^2 - 2t^3$

$= t^3 \left(\frac{3}{t} - 2 \right)$.

As $t \to \infty$, $\frac{3}{t} - 2 \to -2$, and so $f(p(t)) \to -\infty$.

So clearly the set of values of f on $\mathbb{R}^3$ is not bounded below.

We have

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) & \frac{\partial f}{\partial x_2}(x) & \frac{\partial f}{\partial x_3}(x) \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 - 2x_2 x_3 & 2x_2 - 2x_1 x_3 & 2x_3 - 2x_1 x_2 \end{bmatrix}.$$

Thus $\nabla f(u) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and

$\nabla f(v) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

The Hessian $F(x)$ of f at x is given by

$$F(x) = \begin{bmatrix} 2 & -2x_3 & -2x_2 \\ -2x_3 & 2 & -2x_1 \\ -2x_2 & -2x_1 & 2 \end{bmatrix}.$$

Thus $F(u) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$,

and $F(v) = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$.

$F(u)$ is positive definite and $\nabla f(u) = 0$, and so u is a local minimizer.

For a suitable invertible E_1, we have

$$E_1 F(v) E_1^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{bmatrix},$$

and so $F(v)$ is not positive semidefinite.
But for v to be a local minimizer, it is necessary that $F(v)$ is positive semidefinite.
Thus v is not a local minimizer.

(4) (b) The problem (LP) is in standard form:

$$(LP): \begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0, \end{cases}$$

where $c = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 4 \\ 4 \end{bmatrix}$, $A = \begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Let $\beta = (1, 5)$. Then $A_\beta = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, $A_\nu = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

The initial basic solution is $x_\beta = \bar{b}$, $x_\nu = 0$, where $A_\beta \bar{b} = b$ i.e., $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \bar{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$,

and so $\bar{b} = \frac{1}{4} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \geq 0$. So it is a basic feasible solution

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$ i.e., $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} y = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$,

and so $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The reduced costs of the nonbasic variables are given by $r_\nu = c_\nu - A_\nu^T y$, i.e.,

$$r_\nu = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}.$$

Since $r_{\nu_2} = r_3 = -2 < 0$ and it is the smallest, we make x_3 a new basic variable.

We compute $\bar{a}_3$ using $A_\beta \bar{a}_3 = a_3$, i.e.,

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \bar{a}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \text{ and so } \bar{a}_3 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

So the new basic variable x_3 can increase up to

$$t_{max} = \min \left\{ \frac{\bar{b}_k}{\bar{a}_{\nu_q, k}} : \bar{a}_{\nu_q, k} > 0 \right\}$$

$$= \min \left\{ \frac{5/4}{2}, \frac{3/4}{2} \right\} = \frac{3/4}{2} = \frac{\bar{b}_2}{\bar{a}_{3,2}}.$$

The minimizing index is $p = 2$, and hence $x_{\beta_2} = x_5$ leaves the set of basic variables and $x_{\nu_q} = x_3$ takes its place. So $\beta = (1, \boxed{3})$, and $\nu = (2, 4, 5)$. Hence

$$A_\beta = \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix} \text{ and } A_\nu = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}.$$

We calculate $\bar{b}$ using $A_\beta \bar{b} = b$, i.e.,

$$\begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix} \bar{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

and so $\bar{b} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$.

The simplex multipliers vector y is obtained by solving $A_\beta^T y = c_\beta$, i.e., $\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} y = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, and so

$y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The reduced costs of the nonbasic variables are given by $r_\nu = c_\nu - A_\nu^T y$, i.e.,

$$r_\nu = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

As $r_{23} \geq 0$, the current basic feasible solution is optimal for (LP).

Hence

$$\hat{x} := \begin{bmatrix} 1/2 \\ 0 \\ 3/2 \\ 0 \\ 0 \end{bmatrix} \quad \text{is optimal for (LP).}$$

5. (a). The problem can be rephrased as

$$\begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \leq 0 \\ & g_2(x) \leq 0, \end{cases}$$

where
$$f(x) := x_1^2 + x_2^2 - 4x_1,$$
$$g_1(x) := x_1^2 + 4x_2^2 - 1,$$
$$g_2(x) := -x_1 - 2x_2 + 1.$$

g_2, being linear, is a convex function on $\mathbb{R}^2$

The Hessian $G_1(x)$ of g_1 at x is

$$G_1(x) = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$ which is positive (semi) definite

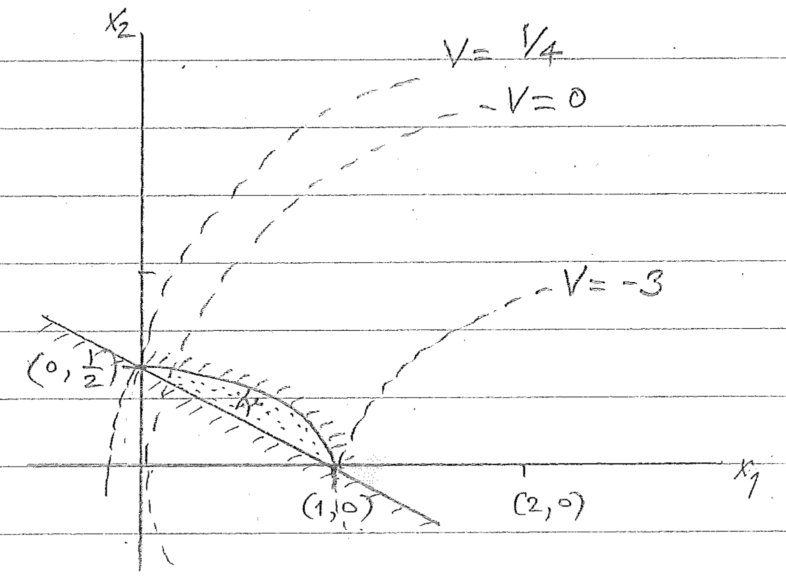
and so g_1 is convex too.

Hence the feasible set $\mathcal{F} := \{x \in \mathbb{R}^2 : g_1(x) \leq 0, g_2 \leq 0\}$ is convex.

The Hessian $F(x)$ of f at x is $F(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ which is positive semidefinite and so f is convex.

So the given problem is convex.

The level sets are circles, since $x_1^2 + x_2^2 - 4x_1 = V$ is the same as $(x_1 - 2)^2 + x_2^2 = V + 4$.



5. (b). We have

$$\nabla f(x) = \begin{bmatrix} 2x_1 - 4 & 2x_2 \end{bmatrix}$$
$$\nabla g_1(x) = \begin{bmatrix} 2x_1 & 8x_2 \end{bmatrix}$$
$$\nabla g_2(x) = \begin{bmatrix} -1 & -2 \end{bmatrix}$$

The problem is regular, since for example with

$$x = \left(\tfrac{1}{2}, \tfrac{3}{8}\right)$$

we have

$$g_1(x) = \frac{1}{4} + 4 \cdot \frac{9}{64} - 1 = \frac{1}{4} + \frac{9}{16} - 1 = \frac{13}{16} - 1 < 0, \text{ and}$$

$$g_2(x) = -\frac{1}{2} - 2 \cdot \frac{3}{8} + 1 = -\frac{1}{2} - \frac{3}{4} + 1 = -\frac{1}{4} < 0.$$

As the problem is a regular convex problem, we know that x is optimal if and only if $\exists\, y \in \mathbb{R}^2$ s.t. the following KKT-conditions are satisfied:

(KKT-1) $\nabla f(x) + y^T \nabla g(x) = 0$, i.e.,

$$\begin{bmatrix} 2x_1 - 4 & 2x_2 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2x_1 & 8x_2 \\ -1 & -2 \end{bmatrix} = 0$$

i.e.,
$$\left.\begin{aligned} 2x_1 - 4 + 2x_1 \cdot y_1 - y_2 &= 0 \\ 2x_2 + 8x_2 \cdot y_1 - 2y_2 &= 0 \end{aligned}\right\}$$

(KKT-2) $g_i \leq 0 \quad \forall\, i = 1, \ldots, m$, i.e.,

$$\begin{cases} x_1^2 + 4x_2^2 \leq 1 \\ x_1 + 2x_2 \geq 1 \end{cases}$$

(KKT-3) $y \geq 0$ i.e., $\begin{cases} y_1 \geq 0 \\ y_2 \geq 0 \end{cases}$

(KKT-4) $y_i\, g_i(x) = 0 \quad \forall\, i = 1, \ldots, m$, i.e.,

$$\begin{cases} y_1\,(x_1^2 + 4x_2^2 - 1) = 0 \\ y_2\,(x_1 + 2x_2 - 1) = 0. \end{cases}$$

With $\hat{x} = (1, 0)$, we have

$$x_1^2 + 4x_2^2 = 1^2 + 4 \cdot 0^2 = 1 \leq 1, \text{ and}$$

$$x_1 + 2x_2 = 1 + 2 \cdot 0 = 1 \geq 1,$$

and so (KKT-2) is satisfied.

Also, $y_1(x_1^2 + 4x_2^2 - 1) = y_1 \cdot (1^2 + 4 \cdot 0^2 - 1) = y_1 \cdot 0 = 0$, and

$y_2(x_1 + 2x_2 - 1) = y_2 \cdot (1 + 2 \cdot 0 - 1) = y_2 \cdot 0 = 0$,

and so (KKT-3) is satisfied.

(KKT-1) becomes: $\begin{cases} 2 \cdot 1 - 4 + 2 \cdot 1 \cdot y_1 - y_2 = 0 \\ 2 \cdot 0 + 8 \cdot 0 \cdot y_1 - 2y_2 = 0 \end{cases}$

i.e., $\begin{cases} -2 + 2y_1 - y_2 = 0 \\ -2y_2 = 0. \end{cases}$

So with $y_2 := 0$ and $y_1 := 1$, these equations are satisfied, i.e., (KKT-1) is satisfied.

Finally as $y_1 = 1 \geq 0$ and $y_2 = 0 \geq 0$, also (KKT-3) is satisfied.

So for $\hat{x} = (1, 0)$, the KKT-conditions are satisfied with $y = (1, 0)$.

Since the problem is regular and convex, we can conclude that $\hat{x} = (1, 0)$ is an optimal solution.