1.(a). Consider the objective function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x) = (x_1-1)^2 + (x_2-2)^2 + e^{x_3}, \qquad x \in \mathbb{R}^3.$$

Its gradient $\nabla f(x)$ at x is given by

$$\nabla f(x) = \begin{bmatrix} 2(x_1-1) & 2(x_2-2) & e^{x_3} \end{bmatrix},$$

and its Hessian $F(x)$ at x is given by

$$F(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & e^{x_3} \end{bmatrix}.$$

As $2>0$ and $e^{x_3}>0$, $F(x)$ is positive definite.

The update equation in Newton's method is:

$$F(x^{(k)})\left(x^{(k+1)} - x^{(k)}\right) = -\left(\nabla f(x^{(k)})\right)^T. \qquad (*)$$

We have

$$\nabla f(0) = \begin{bmatrix} -2 & -4 & 1 \end{bmatrix}$$

$$F(0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(x^{(1)} - 0\right) = -\begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}$$

i.e., $$x^{(1)} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

If $x^{(k)} \xrightarrow{k\to\infty} L$, then as f is $C^{(2)}$, $\nabla f(x^{(k)}) \xrightarrow{k\to\infty} \nabla f(L)$ and $F(x^{(k)}) \xrightarrow{k\to\infty} F(L)$. So from $(*)$, it follows that

$$F(L)(L-L) = -\left(\nabla f(L)\right)^T \quad \text{i.e.,} \quad \nabla f(L) = 0.$$

But $e^{x_3} \neq 0 \quad \forall\, x \in \mathbb{R}^3$ and so $\nabla f(x) \neq 0 \quad \forall\, x \in \mathbb{R}^3$

So $(x^{(k)})_{k \geq 0}$ does not converge.

1.(b). We perform elementary row operations on A to bring it to a "staircase form":

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 5 \\ -3 & -1 & 1 & 3 \end{bmatrix} \xrightarrow{\substack{\text{row } 2 - 2\cdot\text{row } 1 \\ \text{row } 3 - 4\cdot\text{row } 1 \\ \text{row } 4 - 5\cdot\text{row } 1 \\ \text{row } 5 + 3\cdot\text{row } 1}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & -10 & -15 \\ 0 & -5 & -10 & -15 \\ 0 & 5 & 10 & 15 \end{bmatrix}$$

$$\xrightarrow{\substack{-\frac{1}{5}\cdot\text{row } 3 \\ -\frac{1}{5}\cdot\text{row } 4 \\ \frac{1}{5}\,\text{row } 5}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{row } 2 \leftrightarrow \text{row } 5} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{\text{row } 1 - 2\cdot\text{row } 2 \\ \text{row } 3 - \text{row } 2 \\ \text{row } 4 - \text{row } 2}} \begin{array}{c} \begin{matrix} \beta_1 = 1 & \beta_2 = 2 & & \end{matrix} \\ \begin{bmatrix} \left.\begin{matrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{matrix}\right\} U \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \end{bmatrix} \end{array}$$

We have $A_\beta = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 3 \\ 5 & 5 \\ -3 & -1 \end{bmatrix}$.

So a basis for ran A is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 5 \\ -1 \end{bmatrix} \right\}$.

Also,

$$\ker A = \ker U = \left\{ x \in \mathbb{R}^4 : \quad Ux = 0 \right\}$$

$$= \left\{ x \in \mathbb{R}^4 : \quad I x_\beta + U_\nu x_\nu = 0 \right\}$$

$$= \left\{ x \in \mathbb{R}^4 : \quad x_\beta = -U_\nu x_\nu \right\}$$

$$= \left\{ x \in \mathbb{R}^4 : \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \right\}$$

$$x_\nu = e_1 \Rightarrow \quad x_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So $z_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

$$x_\nu = e_2 \Rightarrow \quad x_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

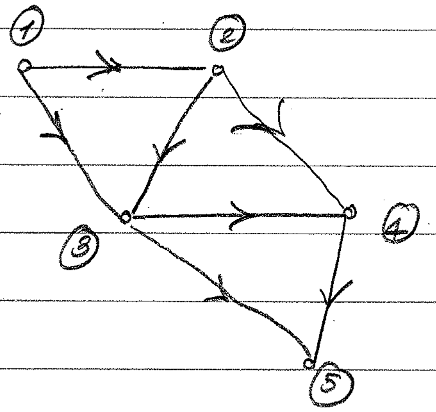
So $z_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

So a basis for $\ker A$ is given by $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2. (a). As A has 5 rows (which add up to 0), number of nodes is 5. We have:

$$A = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array}\begin{array}{c} \begin{array}{ccccccc} (1,2) & (1,3) & (2,3) & (2,4) & (3,4) & (3,5) & (4,5) \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

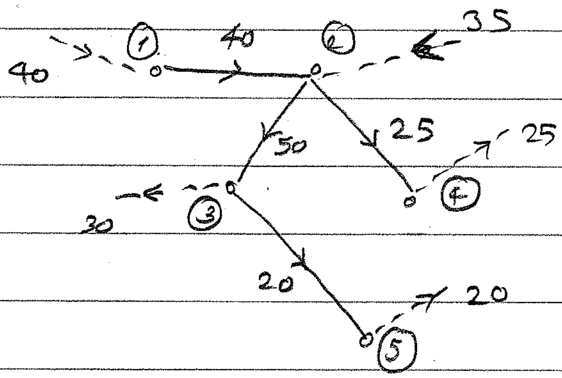
So we arrive at the network:



The given solution

$$x = [\overset{(1,2)}{40} \quad 0 \quad \overset{(2,3)}{50} \quad \overset{(2,4)}{25} \quad 0 \quad \overset{(3,5)}{20} \quad 0]^T$$

corresponds to the spanning tree



① : $40 = 40$

② : $40+35 = 50+25$

③ : $50 = 30+20$

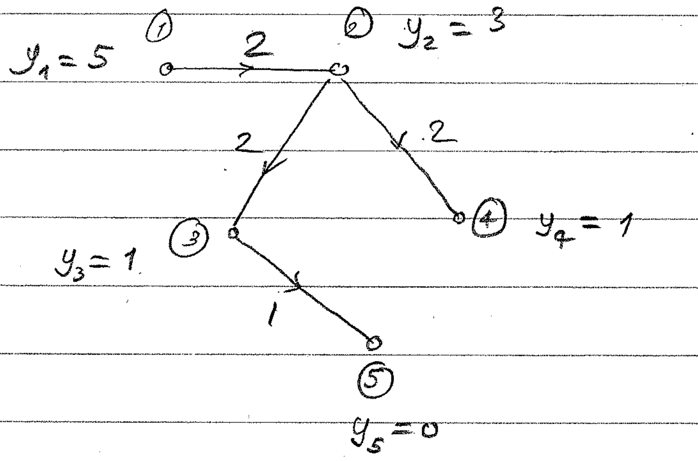
④ : $25 = 25$

⑤ : $20 = 20$

it is ≥ 0, and the flow balance is satisfied at the nodes. So it is a basic feasible solution

We now calculate the simplex multipliers vector using:

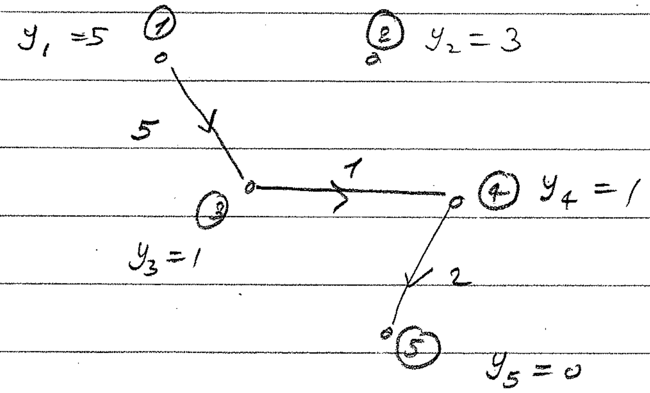
$$y_i - y_j = c_{ij} \quad \text{for the tree edges } (i,j),$$
$$y_m = 0.$$



$$c = \begin{bmatrix} 2 \\ 5 \\ 2 \\ 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \begin{array}{l} \boxed{(1,2)} \\ (1,3) \\ \boxed{(2,3)} \\ \boxed{(2,4)} \\ (3,4) \\ \boxed{(3,5)} \\ (4,5) \end{array}$$

The reduced costs of the nonbasic variables can be found using:

$$r_{ij} = c_{ij} - (y_i - y_j) \quad \text{for the nontree edges } (i,j).$$



Thus:

$$r_{13} = c_{13} - (y_1 - y_3) = 5 - (5 - 1) = 5 - 4 = 1 \geq 0,$$
$$r_{34} = c_{34} - (y_3 - y_4) = 1 - (1 - 1) = 1 - 0 = 1 \geq 0,$$
$$r_{45} = c_{45} - (y_4 - y_5) = 2 - (1 - 0) = 2 - 1 = 1 \geq 0.$$

As all the r_{ij}'s are ≥ 0, the current basic feasible solution is optimal.

2.(b). The dual to

$$(P): \begin{cases} \min. & c^T x \\ s.t. & Ax = b \\ & x \geq 0 \end{cases}$$

is given by

$$(D): \begin{cases} \max. & b^T y \\ s.t. & A^T y \leq c. \end{cases}$$

In our case, this is given by

$$(D): \begin{cases} \text{maximize} & 40y_1 + 35y_2 - 30y_3 - 25y_4 - 20y_5 \\ s.t. & y_1 - y_2 \leq 2 \\ & y_1 - y_3 \leq 5 \\ & y_2 - y_3 \leq 2 \\ & y_2 - y_4 \leq 2 \\ & y_3 - y_4 \leq 1 \\ & y_3 - y_5 \leq 1 \\ & y_4 - y_5 \leq 2. \end{cases}$$

3.(a). For some suitable invertible E_1, we have

$$E_1 H E_1^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2-\frac{1}{2} & -1-\frac{1}{2} \\ 0 & -1-\frac{1}{2} & 2-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix}.$$

With a suitable invertible E_2, we have

$$E_2 E_1 H E_1^T E_2^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As $2 \geqslant 0$, $3/2 \geqslant 0$, and $0 \geqslant 0$, it follows that H is positive semidefinite.

(Alternately, for $x \in \mathbb{R}^3$,

$$x^T H x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1$$
$$= (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \geqslant 0.)$$

3.(b). We have

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1 = x^T \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix} x$$

and so the problem can be rewritten as

$$\begin{cases} \text{minimize} & \frac{1}{2} x^T H x + c^T x + c_0 \\ \text{s.t.} & Ax = b \end{cases}$$

where

$$H = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } c_0 = 0.$$

The Lagrangian method equations are:

$$H\hat{x} - A^T u = -c \quad \text{and}$$

$$A\hat{x} = b.$$

Clearly $A\hat{x} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = b$, and so

$\hat{x}$ is feasible.

Moreover, $H\hat{x} + c = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 2-1-1 \\ -1+2-1 \\ -1-1+2 \end{bmatrix} = 0 = A^T \cdot 0$$

and so with $u := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$, we have also that $H\hat{x} - A^T u = -c$.

As H is positive semidefinite, we know that $\hat{x}$ is optimal if and only if $\exists u$ s.t. $\begin{cases} H\hat{x} - A^T u = c \text{ and} \\ A\hat{x} = b. \end{cases}$

So $\hat{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an optimal solution.

3.(c) $f : C \longrightarrow \mathbb{R}$ is said to be a <u>convex function</u> if $\forall x_1, x_2 \in C$ and $\forall t \in (0,1)$
$$f((1-t)x_1 + t x_2) \leq (1-t) f(x_1) + t f(x_2)$$

3.(d). $(\mathbb{R}^2)^\circ = \mathbb{R}^2 \neq \phi$ and so φ is convex if and only if its Hessian $\Phi(x)$ at each $x \in \mathbb{R}^2$ is positive semidefinite.

We have the gradient of φ at x is
$$\nabla \varphi(x) = \begin{bmatrix} \dfrac{x_1}{\sqrt{x_1^2 + x_2^2}} & \dfrac{x_2}{\sqrt{x_1^2 + x_2^2}} \end{bmatrix}$$ and

the Hessian of φ at x is given by

$$\Phi(x) = \begin{bmatrix} \dfrac{1}{\sqrt{x_1^2+x_2^2}} - \dfrac{x_1 \cdot x_1}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2}} & - \dfrac{x_1 \cdot x_2}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2}} \\[2ex] \dfrac{- x_2 \cdot x_1}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2}} & \dfrac{1}{\sqrt{x_1^2+x_2^2}} - \dfrac{x_2 \cdot x_2}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2}} \end{bmatrix}$$

$$= \frac{1}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2}} \begin{bmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{bmatrix}$$

$$= \frac{1}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2}} \underbrace{\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}}_{= v} \underbrace{\begin{bmatrix} x_2 & -x_1 \end{bmatrix}}_{= v^T}.$$

So if $\omega \in \mathbb{R}^2$, $$\omega^T \Phi(x) \omega = \frac{1}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2}} \omega^T v \, v^T \omega$$

$$= \frac{1}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2}} (\omega^T v)^2 \geq 0.$$

So $\Phi(x)$ is positive semidefinite.

Hence φ is convex.

4. (a). We first calculate candidates for local minimizers

$$\nabla f(x) = [4x_1^3 - 12x_2 \quad -12x_1 + 4x_2^3]$$

Thus $\nabla f(x) = 0 \iff \left\{ \begin{array}{l} 4x_1^3 = 12x_2 \text{ and} \\ 4x_2^3 = 12x_1 \end{array} \right\}$

$$\iff \left(x_1^3 = 3x_2 \text{ and } x_1 = \frac{1}{3} x_2^3 \right) \quad (*)$$

If $(*)$ holds, then $\frac{1}{3^3} x_2^9 = 3x_2$, i.e., $x_2(x_2^8 - 3^4) = 0$.

So $x_2 = 0$ or $x_2^2 = 3$ i.e., $x_2 \in \{0, \sqrt{3}, -\sqrt{3}\}$.

If $x_2 = 0$, $x_1 = 0$; if $x_2 = \sqrt{3}$, $x_1 = \sqrt{3}$; if $x_2 = -\sqrt{3}$, $x_1 = -\sqrt{3}$.

So if $(*)$ holds, then $x \in \{(0,0), (\sqrt{3}, \sqrt{3}), (-\sqrt{3}, -\sqrt{3})\}$

Conversely, if $x \in \{(0,0), (\sqrt{3}, \sqrt{3}), (-\sqrt{3}, -\sqrt{3})\}$, then $(*)$ holds.

The Hessian $F(x)$ of f at x is given by

$$F(x) = \begin{bmatrix} 12x_1^2 & -12 \\ -12 & 12x_2^2 \end{bmatrix} = 12 \begin{bmatrix} x_1^2 & -1 \\ -1 & x_2^2 \end{bmatrix}.$$

Thus $F(0,0) = 12 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, which is not positive semidefinite.

So $(0,0)$ is <u>not</u> a local minimizer.

On the other hand,

$$F(\sqrt{3}, \sqrt{3}) = F(-\sqrt{3}, -\sqrt{3}) = 12 \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

and for a suitable invertible E_1,

$$E_1 \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} E_1^T = \begin{bmatrix} 3 & 0 \\ 0 & 3 - \frac{1}{3} \end{bmatrix}.$$

As $3 > 0$ and $3 - 1/3 > 0$, $F(\sqrt{3}, \sqrt{3})$, $F(-\sqrt{3}, -\sqrt{3})$ are positive definite.

Hence $(\sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, -\sqrt{3})$ are local minimizers.

So the only local minimizers of f are

$$(\sqrt{3}, \sqrt{3}) \text{ and } (-\sqrt{3}, -\sqrt{3})$$

4.(b) The problem is in standard form

$$\begin{cases} \min. & c^T x \\ \text{s.t.} & Ax = b \\ & x \geqslant 0 \end{cases}$$

with $c := \begin{bmatrix} 3 \\ 2 \\ 4 \\ 6 \end{bmatrix}$, $A := \begin{bmatrix} 6 & 3 & 9 & 7 \end{bmatrix}$, $b = 18$.

$m = \text{rank } A = 1$.

$x = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a basic feasible solution corresponding to the basic tuple $\beta = (1)$.

The simplex multiplier vector y is given by

$$A_\beta^T y = c_\beta$$

i.e, $[6]^T y = [3]$, and so $y = 1/2$.

The reduced costs of the non-basic variables are given by $r_\nu = c_\nu - A_\nu^T y$

i.e., $r_\nu = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 9 \\ 7 \end{bmatrix} \frac{1}{2} = \begin{bmatrix} 2 - 3/2 \\ 4 - 9/2 \\ 6 - 7/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 5/2 \end{bmatrix}$.

As $\neg [r_\nu \geqslant 0]$, we can't conclude that the current basic feasible solution is optimal.

Since $r_{\nu_q} = r_{\nu_2} = -\frac{1}{2} < 0$, we make $x_{\nu_q} = x_{\nu_2} = x_3$ the new basic variable. We compute $\bar{a}_{\nu_q} = \bar{a}_3$ using $A_\beta \bar{a}_{\nu_q} = a_{\nu_q}$ i.e, $6 \cdot \bar{a}_3 = a_3 = 9$ and so $\bar{a}_3 = \frac{3}{2}$. Then the new basic variable x_3 can increase upto

$$t_{max} = \min \left\{ \frac{\bar{b}_k}{\bar{a}_{3,k}} : \bar{a}_{3,k} > 0 \right\} = \frac{3}{3/2} = \frac{\bar{b}_1}{\bar{a}_{3,1}}$$

So $x_{\beta_1} = x_1$ leaves the set of basic variables, and x_3 takes its place. Thus $\beta_{new} = (3)$ and $\nu_{new} = (1, 2, 4)$

Hence $A_\beta = [9]$, $A_\nu = [6 \quad 3 \quad 7]$.

We calculate $\bar{b}$ using $A_\beta \bar{b} = b$, i.e.,

$$9\,\bar{b} = 18$$

and so $\bar{b} = 2$.

The simplex multipliers vector is obtained by solving $A_\beta^T y = c_\beta$ i.e., $9y = 4$ and so $y = 4/9$.

The reduced costs of the non-basic variables are given by $r_\nu = c_\nu - A_\nu^T y$

$$= \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 6 \\ 3 \\ 7 \end{bmatrix} \frac{4}{9} = \begin{bmatrix} 3 - \cancel{6}^{2} \cdot \frac{4}{\cancel{9}_3} \\ 2 - \cancel{3}^{1} \cdot \frac{4}{\cancel{9}_3} \\ 6 - 7 \cdot \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{26}{9} \end{bmatrix}.$$

As $r_\nu \geq 0$, the current basic feasible solution is optimal.

So $\hat{x} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ is an optimal solution to (LP)

5. (a) The problem can be rewritten as

$$\begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \\ & g_3(x) \leq 0 \end{cases}$$

where $f, g_1, g_2, g_3 : \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$\left.\begin{aligned} f(x) &= x_1^4 + x_2^4 \\ g_1(x) &= x_1^2 - 1 \\ g_2(x) &= x_2^2 - 1 \\ g_3(x) &= e^{x_1+x_2} - 1. \end{aligned}\right\} \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

The gradients of these functions at x are, respectively:

$$\nabla f(x) = \begin{bmatrix} 4x_1^3 & 4x_2^3 \end{bmatrix}$$

$$\nabla g_1(x) = \begin{bmatrix} 2x_1 & 0 \end{bmatrix}$$

$$\nabla g_2(x) = \begin{bmatrix} 0 & 2x_2 \end{bmatrix}$$

$$\nabla g_3(x) = \begin{bmatrix} e^{x_1+x_2} & e^{x_1+x_2} \end{bmatrix},$$

and their Hessians are, respectively:

$$F(x) = \begin{bmatrix} 12x_1^2 & 0 \\ 0 & 12x_2^2 \end{bmatrix}.$$

$$G_1(x) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G_2(x) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$G_3(x) = \begin{bmatrix} e^{x_1+x_2} & e^{x_1+x_2} \\ e^{x_1+x_2} & e^{x_1+x_2} \end{bmatrix} = e^{x_1+x_2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Clearly $F(x)$ is positive semidefinite since $12x_1^2 \geq 0$ and $12x_2^2 \geq 0$,
$G_1(x)$ " " $2 \geq 0$ and $0 \geq 0$;
$G_2(x)$ " " $0 \geq 0$ and $2 \geq 0$.

Also $e^{x_1+x_2} > 0$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$ and so $G_3(x)$ is positive semidefinite as well.

Since $(\mathbb{R}^2)^\circ \neq \phi$, and $\forall\, x \in \mathbb{R}^2$, $F(x), G_1(x), G_2(x), G_3(x)$ are positive semidefinite, it follows that f, g_1, g_2, g_3 are convex.

So the feasible set

$$\mathcal{F}e = \bigcap_{i=1}^{3} \{x \in \mathbb{R}^2 : g_i(x) \leq 0\}$$

is convex.

The objective function f is also convex.

For $x_0 := \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$,

we have
$$g_1(x_0) = \frac{1}{4} - 1 = -\frac{3}{4} < 0$$
$$g_2(x_0) = 0 - 1 = -1 < 0$$
$$g_3(x_0) = \underbrace{e^{-1/2}}_{\wedge\, 1} - 1 < 0 .$$

As the objective function f is convex,
the feasible set $\mathcal{F}e$ is convex
and $\exists\, x_0 \in \mathbb{R}^2$ s.t. $g_1(x_0), g_2(x_0), g_3(x_0) < 0$,
the problem is a regular convex problem.

5. (b). As the problem is a regular convex problem, we know that x is optimal if and only if $\exists\, y \in \mathbb{R}^3$ s.t. the following KKT-conditions are satisfied:

(KKT-1) : $\nabla f(x) + y^T \nabla g(x) = 0$ i.e.,

$$[4x_1^3 \quad 4x_2^3] + y_1 [2x_1 \quad 0] + y_2 [0 \quad 2x_2] + y_3 [e^{x_1+x_2} \quad e^{x_1+x_2}] = [0 \quad 0]$$

i.e.,
$$\boxed{\begin{aligned} 4x_1^3 + y_1 \cdot 2x_1 + y_3 e^{x_1+x_2} = 0 \\ 4x_2^3 + y_2 \cdot 2x_2 + y_3 e^{x_1+x_2} = 0 \end{aligned}}$$

(KKT-2) $g_i(x) \leq 0 \quad \forall\, i = 1, \dots, m$, i.e.,

$$\boxed{\begin{array}{c} x_1^2 \leq 1 \\ x_2^2 \leq 1 \\ e^{x_1+x_2} \leq 1 \end{array}}$$

(KKT-3) $y \geq 0$ i.e.,

$$\boxed{\begin{array}{c} y_1 \geq 0 \\ y_2 \geq 0 \\ y_3 \geq 0 \end{array}}$$

(KKT-4) $y_i g_i(x) = 0 \quad \forall\, i = 1, \dots, m$, i.e.,

$$\boxed{\begin{array}{c} y_1 (x_1^2 - 1) = 0 \\ y_2 (x_2^2 - 1) = 0 \\ y_3 (e^{x_1+x_2} - 1) = 0 \end{array}}$$

We consider the first the case that $y_1 \neq 0$. Then $x_1 = +1$ or $x_1 = -1$.

$\underline{1.1^\circ}$ $x_1 = 1$. KKT-1 gives $4 + 2\underbrace{y_1}_{>0} + \underbrace{y_3}_{\geq 0} \cdot \underbrace{e^{1+x_2}}_{>0} = 0$,

which is impossible.

$\underline{1.2^\circ}$ $x_1 = -1$. KKT-1 gives $-4 - 2y_1 + y_3 \cdot e^{x_2 - 1} = 0$

i.e., $y_3 \cdot e^{x_2-1} = 4 + 2y_1 > 0$

$(\because y_1 > 0)$

So $y_3 \neq 0$.

Hence $e^{-1+x_2} = 1$

and so $x_2 = 1$.

KKT-1 gives $4 + \underbrace{2y_2}_{\geq 0} + \underbrace{y_3}_{\geq 0} = 0$

which is impossible.

So there is no solution (x, y) to the KKT-conditions when $y_1 \neq 0$.

Since the problem is symmetric in x_1 and x_2, there is no solution when $y_2 \neq 0$.

So $y_1 = y_2 = 0$.

Then KKT-1 gives $4x_1^3 + y_3 \cdot e^{x_1+x_2} = 0$

and $4x_2^3 + y_3 e^{x_1+x_2} = 0$.

As $y_3 . e^{x_1+x_2} \geqslant 0$, it follows that $4x_1^3 \leqslant 0$ and $4x_2^3 \leqslant 0$.

So $x_1 \leqslant 0$ and $x_2 \leqslant 0$. Also $4x_1^3 = 4x_2^3$.

So $x_1 = x_2$.

Now if $y_3 \neq 0$, (KKT-4) gives $e^{2x_1} = 1$ and so $x_1 = 0 = x_2$.

But then KKT-1 gives $y_3 = 0$, a contradiction.

On the other hand if $y_3 = 0$, then KKT-1 gives $x_2 = x_3 = 0$.

So the only solution to the KKT-conditions is $x = (0, 0)$ and $y = (0, 0, 0)$.

Thus $x = (0, 0)$ is optimal.