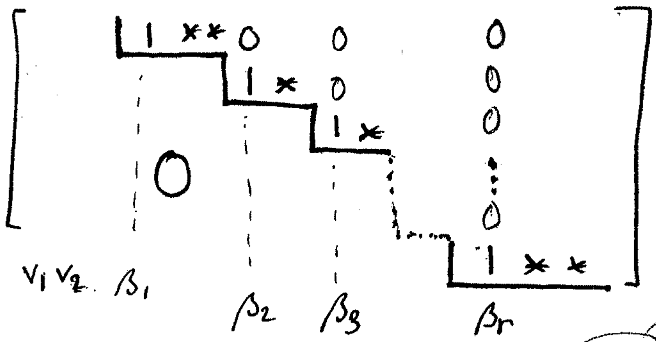
Gauss - Jordan

$$T = PA = \begin{bmatrix} U \\ 0 \end{bmatrix} \qquad U =$$



Skapa Trappstegsmatris T genom radoperationer P (på A).

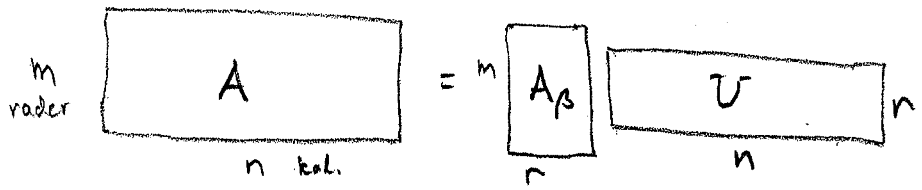
$r = \text{rang}(A)$

$$A_\beta = [\hat{a}_{\beta_1} \dots \hat{a}_{\beta_r}] \qquad U_\beta = [\hat{u}_{\beta_1} \dots \hat{u}_{\beta_r}] = I \qquad U_v = [\hat{u}_{v_1} \dots \hat{u}_{v_{n-r}}]$$

$$A = P^{-1}T = ST = [S_1 \; S_2] \begin{bmatrix} U \\ 0 \end{bmatrix} = S_1 U \qquad \text{där} \quad S = P^{-1}$$

$$\Rightarrow A_\beta = S_1 \underbrace{U_\beta}_{=I} = S_1$$

$$\therefore \; A = A_\beta U \qquad \text{minimal rang faktorisering}$$



A_β har linj. ob kolonner
U har —"— rader

SATS 6.1:
$$R(A) = R(A_\beta)$$
$$N(A) = N(U)$$

Dessutom:
$$R(A^T) = R(U^T)$$
$$N(A^T) = R(A)^\perp = \{y \in \mathbb{R}^m : A_\beta^T y = 0\} = N(A_\beta^T)$$

Ex: Bestäm nollrum och bildrum för A och A^T då

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 2 & 7 & 3 & 1 \\ 2 & 1 & 8 & 1 & 4 \\ -1 & 0 & -3 & 1 & -3 \end{bmatrix}$$

Gör elementära radoperationer för att överföra A på trappstegsform.

$$\begin{array}{r} \\ -1 \\ -2 \\ +1 \end{array} \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 3 & 6 & 2 & 1 \\ 0 & 3 & 6 & -1 & 4 \\ 0 & -1 & -2 & 2 & -3 \end{bmatrix} = A_1 = P_1 A \qquad P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ +1 & 0 & 0 & 1 \end{bmatrix}$$

$\beta_1 = 1$

$$\begin{array}{r} \\ \times 1/3 \\ \\ \\ \end{array} \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2/3 & 1/3 \\ 0 & 3 & 6 & -1 & 4 \\ 0 & -1 & -2 & 2 & -3 \end{bmatrix} = A_2 = P_2 A_1 \qquad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} +1 \\ \\ -3 \\ +1 \end{array} \begin{bmatrix} 1 & 0 & 3 & 5/3 & 1/3 \\ 0 & 1 & 2 & 2/3 & 1/3 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 8/3 & -8/3 \end{bmatrix} = A_3 = P_3 A_2 \qquad P_3 = \begin{bmatrix} 1 & +1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & +1 & 0 & 1 \end{bmatrix}$$

$\beta_2 = 2$

$$\begin{array}{r} \\ \\ \times(-1/3) \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 3 & 5/3 & 1/3 \\ 0 & 1 & 2 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 8/3 & -8/3 \end{bmatrix} = A_4 = P_4 A_3 \qquad P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} -5/3 \\ -2/3 \\ \\ -8/3 \end{array} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A_5 = P_5 A_4 \qquad P_5 = \begin{bmatrix} 1 & 0 & -5/3 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -8/3 & 1 \end{bmatrix}$$

$\beta_3 = 4$

Nu har vi A på trappstegsform:

$$T = A_5 = \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} U \\ 0 \end{bmatrix}$$

$$\text{där} \quad U = \begin{array}{c} \begin{bmatrix} 1 & 0 & 3 & 0 & \boxed{2} \\ 0 & 1 & 2 & 0 & \boxed{1} \\ 0 & 0 & 0 & 1 & \boxed{-1} \end{bmatrix} \\ \begin{array}{ccccc} \beta_1 & \beta_2 & \nu_1 & \beta_3 & \nu_2 \end{array} \end{array} \qquad r = \operatorname{rang}(A) = 3$$

$$T = A_5 = P_5 A_4 = P_5 P_4 A_3 = P_5 P_4 P_3 A_2 = P_5 P_4 P_3 P_2 A_1 = P_5 P_4 P_3 P_2 P_1 A = PA$$

där P är en inverterbar matris.

dvs $A = P^{-1} T$

$$A = A_\beta U \quad \text{där} \quad A_\beta = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_4 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

- $$\mathcal{R}(A) = \mathcal{R}(A_\beta) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- $$\mathcal{N}(A) = \mathcal{N}(A_\beta U) = \mathcal{N}(U) = \{ x \in \mathbb{R}^5 : x_\beta = -U_\delta x_\delta, \; x_\delta \in \mathbb{R}^2 \}$$

$$= \left\{ x \in \mathbb{R}^5 : \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = -\begin{pmatrix} 3 & 2 \\ 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_5 \end{pmatrix}, \; \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} \in \mathbb{R}^2 \right\}$$

$$= \left\{ x \in \mathbb{R}^5 : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -2 \\ -1 \\ 0 \\ +1 \\ 1 \end{pmatrix} \boxed{x_5} \quad \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} \in \mathbb{R}^2 \right\}$$

$$= \operatorname{span}\left\{ \underset{v_1}{\begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} \boxed{-2} \\ \boxed{-1} \\ 0 \\ \boxed{1} \\ \boxed{1} \end{pmatrix}} \right\}$$

<u>Koll:</u> $Av_1 = 0$, $Av_2 = 0$ OK!

- $\mathcal{R}(A^T) = \mathcal{R}(U^T) = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 2 \end{bmatrix}}_{x_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{x_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}_{x_3} \right\}.$

<u>Koll</u>: $x_k^T v_l = 0$

$\mathcal{R}(A^T) \perp \mathcal{N}(A)$.

- $\mathcal{N}(A^T) = \mathcal{N}(A_\beta^T)$

$$A_\beta^T = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 2 & 1 & 0 \\ 1 & 3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 3 & 3 & -1 \\ 0 & 2 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -2/3 \\ 0 & 1 & 1 & -1/3 \\ 0 & 0 & -3 & 8/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2/9 \\ 0 & 1 & 0 & 5/9 \\ 0 & 0 & 1 & -8/9 \end{bmatrix}$$

$$\mathcal{N}(A_\beta^T) = \text{span}\left\{ \begin{bmatrix} -2/9 \\ -5/9 \\ 8/9 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \underbrace{\begin{bmatrix} -2 \\ -5 \\ 8 \\ 9 \end{bmatrix}}_{w_1} \right\}$$

<u>Koll</u>: $A_\beta^T w_1 = 0$

alt. $A^T w_1 = 0$