1.1.1. Linear programming. If the objective function is a linear function and the feasible set is given by a bunch of linear inequalities, then the corresponding optimization problem (1.1) is called *linear programming*. Thus the general linear programming problem has the following form:

$$\begin{cases} \text{minimize} & c^\top x, \\ \text{subject to} & Ax \geq b, \end{cases}$$

where $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are given vectors, and $A \in \mathbb{R}^{m\times n}$ is a given matrix. The inequality "$\geq$" above means that this inequality holds component-wise. Thus there are m scalar inequalities in $Ax \geq b$. This problem is a special case of (1.1), where

$$f(x) = c^\top x \quad \text{and} \quad \mathcal{F} = \{x \in \mathbb{R}^n : Ax \geq b\}.$$

1.1.2. Quadratic optimization. If the objective function is a quadratic function and the feasible set is given by a bunch of linear inequalities, then the corresponding optimization problem (1.1) is called *quadratic optimization*. Thus the general quadratic optimization problem has the following form:

$$\begin{cases} \text{minimize} & \frac{1}{2}x^\top Hx + c^\top x, \\ \text{subject to} & Ax \geq b, \end{cases}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $H \in \mathbb{R}^{n\times n}$ is a symmetric matrix. This problem is a special case of (1.1), where

$$f(x) = \frac{1}{2}x^\top Hx + c^\top x \quad \text{and} \quad \mathcal{F} = \{x \in \mathbb{R}^n : Ax \geq b\}.$$

1.1.3. Nonlinear optimization. The *nonlinear optimization problem* has the following form:

$$\begin{cases} \text{minimize} & f(x), \\ \text{subject to} & g_i(x) \leq 0, \ i = 1, \dots, m, \end{cases}$$

where f and $g_1, \dots, g_m$ are given functions from $\mathbb{R}^n$ to $\mathbb{R}$. These functions will be assumed to be continuously differentiable, and at least one of them will be assumed to be nonlinear (otherwise, we will have a linear programming problem). The feasible set in this case is given by

$$\mathcal{F} = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, \dots, m\}.$$

1.2. Minimum of a subset of $\mathbb{R}$

Definition 1.1. Let S be a subset of $\mathbb{R}$.

(1) An element $u \in \mathbb{R}$ is said to be an *upper bound of* S if for all $x \in S$, $x \leq u$. If the set of all upper bounds of S is not empty, then S is said to be *bounded above*.

(2) An element $l \in \mathbb{R}$ is said to be a *lower bound of* S if for all $x \in S$, $l \leq x$. If the set of all lower bounds of S is not empty, then S is said to be *bounded below*.

Example 1.2.

(1) The set $S = \{x \in \mathbb{R} : 0 \leq x < 1\}$ is bounded above and bounded below. Any real number y satisfying $1 \leq y$ (for instance 1, 2, 100) is an upper bound of S, and any real number z satisfying $z \leq 0$ (for instance 0, -1) is a lower bound of S.

(2) The set $S = \{n : n \in \mathbb{N}\}$ is not bounded above. Although it is bounded below (any real number $x \leq 1$ serves as a lower bound), it has no upper bound, and so it is not bounded above.

(3) The set[1] $S = \{(-1)^n : n \in \mathbb{N}\}$ is bounded above and bounded below. It is bounded above by 1 and bounded below by -1. More generally, any finite set S is bounded.

[1] Note that this set is simply the finite set $\{-1, 1\}$.