KTH Mathematics

SF2812 Applied linear optimization, final exam
Friday January 9 2009 8.00–13.00
Brief solutions

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1. (a) The primal-dual system of equations can be written as

$$x_1 + x_2 = 1, \tag{1a}$$
$$y + s_1 = 1, \tag{1b}$$
$$y + s_2 = 1, \tag{1c}$$
$$x_1 s_1 = \mu, \tag{1d}$$
$$x_2 s_2 = \mu. \tag{1e}$$

where we also implicitly demand $x > 0$ and $s > 0$. We conclude from (1b) and (1c) that $s_1 = s_2$. Hence, (1d) and (1e) give $x_1 = x_2 = 1/2$. Thus, $s_1 = s_2 = 2\mu$. Finally, $y = 1 - 2\mu$.

In summary,

$$x(\mu) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad y(\mu) = 1 - 2\mu, \quad s(\mu) = \begin{pmatrix} 2\mu \\ 2\mu \end{pmatrix}.$$

(b) Letting $\mu \to 0$ gives

$$x = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad y = 1, \quad s = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It is straightforward to very that $Ax = b$, $x \ge 0$, $A^T y + s = c$, $s \ge 0$. Consequently, optimality holds.

(c) The given optimal solution to (LP) is not a basic feasible solution. Hence, it would not have been given by the simplex method.

2. (See the course material.)

3. (a) Lagrangian relaxation for $u = (1\ 1)^T$ gives

$$\begin{array}{ll} \min & -5x_1 - 8x_2 - 9x_3 - 9x_4 \\ \text{subject to} & -3x_1 - 4x_2 - 5x_3 - 6x_4 \ge -9, \\ & x \ge 0, \ x \text{ integer.} \end{array}$$

This problem is equivalent to the given knapsack problem. The optimal solution is $x(1) = (0\ 1\ 1\ 0)^T$.

(b) Insertion of the optimal solution $x(1)$ in the relaxed constraints gives a subgradient according to

$$\begin{pmatrix} -2 + x_1(1) + x_2(1) + x_3(1) \\ -2 + x_2(1) + x_3(1) + x_4(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the subgradient is zero, it follows that $u = (1\ 1)^T$ is optimal to the dual problem. In addition, it follows that the duality gap is zero and $x(1)$ is optimal to (IP).

4. The values of b_1 and b_2 must be such that $A\widehat{x} = b$, which gives $b_1 = 6$ and $b_2 = 10$. For these values of b_1 and b_2, the given $\widehat{x}$ is feasible.

The given $\widehat{x}$ is not a basic feasible solution. In order for $\widehat{x}$ to be optimal, there cannot be a basic feasible solution with lower objective function value. To find a basic feasible solution, we may compute directions in the null space of A_+, and successively add constraints. The v given in the hint is such that $A_+ v_+ = 0$, $v_0 = 0$. Hence, if $\widehat{x}$ is optimal, it must hold that $c^T v = 0$. This implies that $c_1 = 3$. If we compute the maximum value of α such that of $\widehat{x} + \alpha v \ge 0$, we obtain $\alpha_{\max} = 1$. The point $\widehat{x} + \alpha_{\max} v$ has one more active constraint, and is in fact a basic feasible solution, with $x_1 = 4$ and $x_3 = 2$ as basic variables. The simplex multipliers are given by $B^T y = c_B$, i.e.,

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix},$$

which gives $y = (5\ \ -2)^T$. The reduced costs are now given by $s = c - A^T y = (0\ 0\ 0\ c_4 + 3)^T$. Consequently, $s \ge 0$ if $c_4 \ge -3$. As the basic variables are strictly positive, it follows that the basic feasible solution is not optimal if $c_4 < -3$. Hence, we conclude that $\widehat{x}$ is optimal if and only if $b_1 = 6$, $b_2 = 10$, $c_1 = 3$ and $c_4 \ge -3$.

5. (a) For the given cut patterns, we obtain

$$B = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad x_B = B^{-1} b = \begin{pmatrix} 1.875 \\ 7.5 \\ 6.25 \end{pmatrix}, \quad y = B^{-T} e = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{1}{2} \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $\lambda \ge 0$ no slack variables enters the basis.

We obtain the subproblem

$$\begin{array}{ll} 1 - \ \frac{1}{8}\text{maximize} & 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \\ \text{subject to} & 3\alpha_1 + 4\alpha_2 + 5\alpha_3 \le 11, \\ & \alpha_i \ge 0, \ \text{integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is zero. Hence, the linear program has been solved, and its optimal value is 15.625.

(b) If we round up, we obtain $\widetilde{x} = (2\ 8\ 7)$, which is at most one roll away from optimality, since we use 17 rolls, and the lower bound from the LP relaxation may be rounded up to 16. In fact, we may try to decrease each component of $\widetilde{x}$ by one, which is feasible for the third component, and we obtain an optimal solution $\widehat{x} = (2\ 8\ 6)$.

(Note that this is very special. In general one can not expect to obtain an optimal integer solution in this way.)