KTH Mathematics

SF2812 Applied linear optimization, final exam
Monday October 19 2009 14.00–19.00
Brief solutions

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1. (See the course material.)

2. Let z denote the integer variable and let x denote the continuous variables.

At node 0, the LP relaxation of the original problem is solved. Let x_0, z_0 denote an optimal solution to this linear program. If z_0 is integer, we have solved the original problem at node 0, i.e., the original problem.

If z_0 is noninteger, there will be two new nodes in the search tree, one node (node 1) with the additional constraint $z \geq \text{ceil}(z_0)$, and one node (node 2) with the constraint $z \leq \text{floor}(z_0)$, where floor means rounding down to the nearest integer and ceil means rounding up to the nearest integer.

At node 1, the LP relaxation is solved. If this LP is infeasible, the problem at node 1 is infeasible and the node is fathomed. Otherwise, let x_1, z_1 denote an optimal solution. If $z_1 = \text{ceil}(z_0)$, then the problem at node 1 has been solved, and the node is fathomed. If $z_1 > \text{ceil}(z_0)$, then the constraint $z \leq \text{ceil}(z_0)$ is inactive at x_1, z_1. Hence, x_1, z_1 is optimal to the LP relaxed problem of node 0 as well. By assumption that this LP had a unique optimal solution, this cannot happen.

The same argument can be applied to node 2, with the constraint $z \geq \text{ceil}(z_0)$ replaced by $z \leq \text{floor}(z_0)$.

Hence, it takes at most three nodes.

(The assumption about unique LP solutions is merely to avoid some technicalities. Note that if $z_1 > \text{ceil}(z_0)$, then

$$\begin{pmatrix} x_0 \\ z_0 \end{pmatrix} + \frac{\text{ceil}(z_0) - z_0}{z_1 - z_0} \begin{pmatrix} x_1 - x_0 \\ z_1 - z_0 \end{pmatrix}$$

is an optimal solution to the LP relaxation at node 0 where the z component is integer, $\text{ceil}(z_0)$. Hence, the integer program has been solved. The result for node 2 is analogous.)

3. The basis corresponding to $\widetilde{y}$ and $\widetilde{s}$ is $\mathcal{B} = \{1, 4\}$. Let $y = \widetilde{y}$ and $s = \widetilde{s}$. It is straightforward to verify that $B^T y = c_B$ and $s = c - A^T y \geq 0$. Hence, y and s are dual feasible. The basic variables are given by

$$\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

which gives $x_1 = -1/3$, $x_4 = 7/3$. As $x_1 < 0$, the dual solution is not optimal. Consequently, since $x_1 < 0$, x_1 becomes nonbasic, and as x_1 is the first basic variable, the step in the y-direction is given by

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

which gives $q_1 = -1/3$, $q_2 = 1/3$. With $y \leftarrow y + \alpha q$, dual feasibility requires $s \leftarrow s + \alpha\eta$, with $A^T q + \eta = 0$ and $s + \alpha\eta \geq 0$. Consequently, the nonnegativity of s requires $s - \alpha A^T q \geq 0$, i.e.,

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The maximum value of α is given by $\alpha_{\max} = 3/2$ making component 2 of $s - \alpha A^T q$ zero, so that the new basis becomes $\mathcal{B} = \{2, 4\}$. The basic variables are given by

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

which gives $x_2 = 1/2$, $x_4 = 3/2$. As $x \geq 0$, an optimal solution has been obtained. Together with $y + \alpha_{\max} q$ and $s - \alpha_{\max} A^T q$ the primal and dual optimal solutions are given by

$$x = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix}, \quad y = \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} \frac{3}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

4. The suggested initial extreme points $v_1 = (0 \;\; 2 \; -1 \;\; 0)^T$ and $v_2 = (0 \;\; 2 \;\; 0 \;\; 2)^T$ give the initial basis matrix

$$B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (1 \; 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1 \ c^T v_2) = (-8 \ -6)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -8 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}.$$

By forming $c^T - y_1 A = (1 \ -3 \ 0 \ 0)$ we obtain the subproblem

$$\begin{array}{lll} 6 + & \text{minimize} & x_1 - 3x_2 \\ & \text{subject to} & -2 \le 2x_1 - x_2 \le 2, \\ & & -2 \le 2x_1 + x_2 \le 2, \\ & & -2 \le 2x_3 - x_4 \le 2, \\ & & -2 \le 2x_3 + x_4 \le 2. \end{array}$$

The resulting optimal solution is given by $x_1 = 0$ and $x_2 = 2$. The values of x_3 and x_4 can be chosen as any extreme point of S. Consequently, the optimal value of the subproblem is zero, and the original problem has been solved. The optimal solution x^* is given by

$$x^* = v_1\alpha_1 + v_3\alpha_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} \frac{1}{2} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 0 \\ 2 \\ -\frac{1}{2} \\ 1 \end{pmatrix}.$$

5. (a) For a given $u \ge 0$, we obtain

$$\begin{array}{llll} \varphi(u) = -Tu + & \text{minimize} & \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij} + ut_{ij})x_{ij} & \\ & \text{subject to} & \sum_{j=1}^{n} x_{ij} = a_i, & i = 1,\dots,m, \\ & & \sum_{i=1}^{m} x_{ij} = b_j, & j = 1,\dots,n, \\ & & x_{ij} \in \{0,1\}, & i = 1,\dots,m, \ j = 1,\dots,n. \end{array}$$

This is a transportation problem, which may be solved as a linear program. A subgradient to $\varphi(u)$ at u is given by

$$-T + \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij}(u).$$

(b) We obtain

$$\begin{array}{lll} \varphi(u) = \sum_{i=1}^{m} a_i v_i + \sum_{j=1}^{n} b_j w_j + & \text{minimize} & \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij} - v_i - w_j)x_{ij} \\ & \text{subject to} & \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij} \le T, \\ & & x_{ij} \in \{0,1\}, \ i = 1,\dots,m, \ j = 1,\dots,n, \end{array}$$

This is a knapsack problem.

A subgradient to $\varphi(v,w)$ at (v,w) is given by

$$\begin{pmatrix} a_1 - \sum_{j=1}^n x_{1j}(v,w) \\ \vdots \\ a_m - \sum_{j=1}^n x_{mj}(v,w) \\ b_1 - \sum_{i=1}^m x_{i1}(v,w) \\ \vdots \\ b_n - \sum_{i=1}^m x_{in}(v,w) \end{pmatrix}$$

(c) The first relaxation gives a transportation problem, where the integer requirement on x may be relaxed without altering the problem. Hence, the bound from the first relaxation is identical to the bound from an LP-relaxation. The second requirement gives a knapsack problem. Since the Lagrangian dual is always at least as good as the LP relaxation, we therefore expect the dual based on the second relaxation to give a better bound.