KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday October 21 2010 14.00–19.00
Brief solutions

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1. (See the course material.)

2. As $\widehat{x}_j > 0$, $j = 1, 2, 3, 5$, the active constraints at $\widehat{x}$ are given by

$$\begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \\ \widehat{x}_4 \\ \widehat{x}_5 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \\ 16 \\ 0 \end{pmatrix}.$$

These constraints remain active for $\widehat{x} + \alpha p$, where p satisfies

$$\begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From the given hint we obtain $p = (1 \ \ -1 \ \ 2 \ \ 0 \ \ -2)^T$. The additional requirement $\widehat{x} + \alpha p \geq 0$ gives

$$\begin{pmatrix} 3 \\ 5 \\ 2 \\ 0 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ -2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that $\widehat{x} + \alpha p \geq 0$ for $-1 \leq \alpha \leq 1$. In addition, it holds that $c^T p = 0$, so that $\widehat{x} + \alpha p$ has the same objective function value as $\widehat{x}$ for all α. By taking the limiting values of α, we get two new points at which five constraints are active, namely

$$x^{(1)} = \widehat{x} - p = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \quad x^{(2)} = \widehat{x} + p = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

As there are five active constraints at these points, we expect them to be basic feasible solutions. By assuming that x_1, x_2 and x_5 are basic variables, we may compute y and s from $B^T y = c_B$, $s = c - A^T y$, i.e.,

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

with solution $y = (-1\ 1\ 0)^T$, so that $s = c - A^T y = (0\ 0\ 0\ 1\ 0)^T$. As $s \geq 0$, we have verified optimality of $x^{(1)}$, and hence $\widehat{x}$ and $x^{(2)}$ are optimal as well. It is straightforward to verify that $x^{(2)}$ is also a basic feasible solution at which x_1, x_2 and x_3 are basic variables.

Comment: Note that *all* active constraints at $\widehat{x}$ must be considered when contructing p. It is *not* sufficient to require $Ap = 0$. Assume, for example, that $c = 0$. Then all three directions suggested in the hint satisfy $Ap = 0$, $c^T p = 0$. However, as $\widehat{x}_4 = 0$, we must in addition have $p_4 = 0$ to be able to take steps in both directions along p, and obtain two new points with one more contraint active. The first and third directions given in the hint only allow a nonzero step in one direction, and the active constraint $x_4 = 0$ becomes inactive. Hence, maximum steps along these directions would in each case create a new point with four active constraints. Then, we would face exactly the same situation as at $\widehat{x}$, and no progress towards a basic feasible solution would have been made.

3. (a) If we introduce dual variables y, associated with $Ax - b = 0$, and v_i, associated with $p_i T_i x + p_i W_i u_i - p_i h_i = 0$, $i = 1, \dots, N$, the dual may be written as

$$(D_p) \quad \begin{array}{ll} \text{maximize} & b^T y + \sum_{i=1}^{N} p_i h_i^T v_i \\ \text{subject to} & A^T y + \sum_{i=1}^{N} p_i T_i^T v_i \leq c, \\ & p_i W_i^T v_i \leq p_i d_i, \quad i = 1, \dots, N. \end{array}$$

(Note that the constraint $p_i W_i^T v_i \leq p_i d_i$ can be simplified to $W_i^T v_i \leq d_i$.)

(b) As $d_i \geq 0$, $i = 1, \dots, N$, it follows that $y = y^*$, $v_i = 0$, $i = 1, \dots, N$, gives

$$A^T y + \sum_{i=1}^{N} p_i T_i^T v = A^T y^* \leq c,$$
$$p_i W^T v_i = 0 \leq p_i d_i, \quad i = 1, \dots, N.$$

Hence, $y = y^*$, $v_i = 0$, $i = 1, \dots, N$, gives a feasible solution to (D_p).

(c) Problem (P) is a relaxation of (P_p). Both requirements for relaxation are fulfilled: (i) The feasible region of (P_p) is a subset of the feasible region of (P), and (ii) $p_i d_i^T u_i \geq 0$, $i = 1, \dots, N$, holds in (P_p) as $p_i > 0$, $d_i \geq 0$ and $u_i \geq 0$. Consequently, it follows that $\text{optval}(P_p) \geq \text{optval}(P)$.

(d) We have $y = y^*$, $v_i = 0$, $i = 1, \dots, N$ feasible to (D_P) with objective function value $b^T y^*$. As (P_p) is feasible, it follows that $\text{optval}(D_p)$ is bounded, so that $\text{optval}(P_p) = \text{optval}(D_p) \geq b^T y^* = \text{optval}(D) = \text{optval}(P)$, where strong duality for linear programming has been used.

4. (a) The dual objective $\varphi(v)$ is the optimal solution of

$$\begin{aligned} &\text{minimize } -x_1 - 3x_3 - x_4 + v_1(x_1 + x_2 - 1) + v_2(x_3 + x_4 - 1) \\ &\text{subject to } 4x_1 + 5x_2 + 6x_3 + 7x_4 \le 10,\ x_j \in \{0,1\},\ j = 1,\dots,4, \\ = \ & -v_1 - v_2 - \text{maximize } (1 - v_1)x_1 - v_1 x_2 + (3 - v_2)x_3 + (1 - v_2)x_4 \\ &\qquad \text{subject to } 4x_1 + 5x_2 + 6x_3 + 7x_4 \le 10, \\ &\qquad x_j \in \{0,1\},\ j = 1,\dots,4. \end{aligned}$$

In particular, for $v = \widehat{v}$, we obtain

$$\begin{aligned} \varphi(\widehat{v}) = -3 - \text{maximize } & -x_2 + x_3 - x_4 \\ \text{subject to } & 4x_1 + 5x_2 + 6x_3 + 7x_4 \le 10, \\ & x_j \in \{0,1\},\ j = 1,\dots,4. \end{aligned}$$

It follows that $x_2 = 0$ and $x_4 = 0$ in all optimal solutions, since the corresponding objective function coefficients are negative in the maximization problem. Hence, we obtain two optimal solutions, $x^{(1)}(\widehat{v}) = (0\ 0\ 1\ 0)^T$ and $x^{(2)}(\widehat{v}) = (1\ 0\ 1\ 0)^T$ with $\varphi(\widehat{v}) = -4$.

(b) We obtain two subgradients $s^{(1)}$ and $s^{(2)}$ to φ at $\widehat{v}$ by evaluating the relaxed constraints with reversed sign at $x^{(1)}(\widehat{v})$ and $x^{(2)}(\widehat{v})$ respectively, as

$$\begin{aligned} s^{(1)} &= -\begin{pmatrix} 1 - x_1^{(1)}(\widehat{v}) - x_2^{(1)}(\widehat{v}) \\ 1 - x_3^{(1)}(\widehat{v}) - x_4^{(1)}(\widehat{v}) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \\ s^{(2)} &= -\begin{pmatrix} 1 - x_1^{(2)}(\widehat{v}) - x_2^{(2)}(\widehat{v}) \\ 1 - x_3^{(2)}(\widehat{v}) - x_4^{(2)}(\widehat{v}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

(c) As $s^{(2)} = 0$, it follows that $\widehat{v}$ is optimal to the dual problem.

5. As $W/w_1 = 4\frac{1}{3}$, the cut patterns with w_1-rolls only is given by $(4\ 0\ 0)^T$. The two other analgous cut patterns are given by $(0\ 2\ 0)^T$ and $(0\ 0\ 2)^T$.

Consequently, we obtain $A_1 = (4\ 0\ 0)^T$, $A_2 = (0\ 2\ 0)^T$ and $A_3 = (0\ 0\ 2)^T$, so that

$$B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad x_B = B^{-1}b = \begin{pmatrix} 10 \\ 45 \\ 20 \end{pmatrix}, \quad y = B^{-T}e = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $y \ge 0$ no slack variables enter the basis.

We obtain the subproblem

$$\begin{aligned} 1 - \ \tfrac{1}{4}\text{maximize } & \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ \text{subject to } & 3\alpha_1 + 5\alpha_2 + 7\alpha_3 \le 14, \\ & \alpha_i \ge 0,\ \text{integer},\quad i = 1,2,3. \end{aligned}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal solutions to the subproblem are given by $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 0$, and $\alpha_1 = 3$, $\alpha_2 = 1$, $\alpha_3 = 0$, with optimal value -1/4. As suggested in the statement, we let $A_4 = (1\ 2\ 0)^T$ with

$$p_B = -B^{-1}A_4 = \begin{pmatrix} -\frac{1}{4} \\ -1 \\ 0 \end{pmatrix}.$$

The minimum ratio occurs for $\alpha_{\max} = 40$, when the first basic variable becomes zero, so that x_1 leaves the basis. Hence,

$$B = \begin{pmatrix} A_4 & A_2 & A_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

so that

$$x_B = \begin{pmatrix} x_4 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 40 \\ 5 \\ 20 \end{pmatrix}, \quad y = y = B^{-T}e = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $y \geq 0$ no slack variables enter the basis.

We obtain the subproblem

$$\begin{array}{lll} 1 - & \frac{1}{2}\text{maximize} & \alpha_2 + \alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 7\alpha_3 \leq 14, \\ & & \alpha_i \geq 0, \text{ integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is zero. Hence, the linear program has been solved.

However, as it so happens that x_B is integer valued, the original problem has been solved as well. An optimal solution to the original problem is thus given by cutting 40 W-rolls according to cut pattern $(1\ 2\ 0)^T$, 5 W-rolls according to cut pattern $(0\ 2\ 0)^T$ and 20 rolls according to cut pattern $(0\ 0\ 2)^T$.

(Note that this is very special. In general x_B will not take on integer values.)