KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday January 13 2011 8.00–13.00
Brief solutions

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1. (a) A basic feasible solution is a feasible solution which is uniquely defined by the active constraints. As $\widehat{x}_j > 0$, $j = 1, 2, 3$, the active constraints at $\widehat{x}$ are given by

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \\ \widehat{x}_4 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 0 \end{pmatrix}.$$

As there are three active constraints at $\widehat{x}$, and $\widehat{x}$ has dimension four, $\widehat{x}$ cannot be uniquely defined by the active constraints. Hence, $\widehat{x}$ is not a basic feasible solution.

(b) The constraints that are active at $\widehat{x}$ remain active for $\widehat{x} + \alpha p$, where p satisfies

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is straightforward to find such a p. If we let $p_1 = 1$, then $p = (1 \;\; -2 \;\; 3 \;\; 0)^T$. This means that p is uniquely determined up to a multiple scalar. Since $c^T p = 5$, we conclude that by finding the minimum value of α such that $\widehat{x} + \alpha p \geq 0$, we obtain a new feasible point with four active constraints. This gives

$$\begin{pmatrix} 5 \\ 2 \\ 3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that the minimum value is $\alpha = -1$. Since p is unique up to a scalar, the nullspace of the active constraints at $\widehat{x}$ has dimension one. At $\widehat{x} - p$, we add one more constraint, which is not linearly dependent. Hence, $\widehat{x} - p$ will be uniquely determined by the active constraints, i.e., a basic feasible solution. Consequently, we may let $\widetilde{x} = \widehat{x} - p = (4 \;\; 4 \;\; 0 \;\; 0)^T$.

(c) Since $\widetilde{x}$ is a basic feasible solution, we may apply the primal simplex method starting at $\widetilde{x}$.

Then, x_1 and x_2 are basic variables. We may compute y and s from $B^T y = c_B$, $s = c - A^T y$, i.e.,

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix},$$

with solution $y = (-5/3 \;\; 1/3)^T$, so that $s = c - A^T y = (0 \;\; 0 \;\; 5/3 \;\; -1/3)^T$. As $s_4 < 0$, we take a step along p given by $p_4 = 1$ and

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which gives $p_1 = -2/3$ and $p_2 = 1/3$. By setting $x_B + \alpha p_B \geq 0$, we obtain $\alpha_{\max} = 6$. Consequently, new basic variables are $x_2 = 6$ and $x_4 = 6$.
We may compute y and s from $B^T y = c_B$, $s = c - A^T y$, i.e.,

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix},$$

with solution $y = (-3/2 \;\; 0)^T$, so that $s = c - A^T y = (1/2 \;\; 0 \;\; 3/2 \; 0)^T$. As $s \geq 0$, we have an optimal solution. Hence, $x = (0 \;\; 6 \;\; 0 \;\; 6)^T$ is optimal to (LP).

2. (See the course material.)

3. (a) Insertion of numercial values gives $A\widehat{x} = b$, $A^T\widehat{y} + s = c$. In addition, $\widehat{x} \geq 0$, $\widehat{s} \geq 0$ and $\widehat{x}_j \widehat{s}_j = 0$, $j = 1, \ldots, 5$. Hence, the solutions are optimal to the primal and dual problems, respectively.

(b) The solution given by $\widehat{x}$ corresponds to x_1 and x_2 being basic variables. Since $\widehat{s}_3 = 0$, it follows that x_3 may enter the basis without changing the value of the objective function. Consequently, optimality is preserved. The corresponding direction is given by $p_3 = 1$ and

$$\begin{pmatrix} 6 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = - \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which gives $p_1 = -1/2$ and $p_2 = 1$. By setting $x_B + \alpha p_B \geq 0$, we obtain $\alpha_{\max} = 2$. Consequently, new basic variables are $x_2 = 3$ and $x_3 = 2$.
We may compute y and s from $B^T y = c_B$, $s = c - A^T y$, i.e.,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix},$$

with solution $y = (2 \;\; -1)^T$, so that $s = c - A^T y = (0 \;\; 0 \;\; 0 \; 3 \;\; 3)^T$. As $s \geq 0$, we have an optimal solution. In addition, since $s_1 = 0$ but $s_4 > 0$ and $s_5 > 0$, it follows that it is only x_1 that may enter the basis again without increasing the objective function value. This would give us $\widehat{x}$ back. Consequenty, there are only two optimal basic feasible solutions, $\widehat{x}$ and $(0 \;\; 3 \;\; 2 \;\; 0 \;\; 0)^T$. Therefore, the set of optimal solutions is given by the set of convex combinations of these points, i.e.,

$$\left\{ (1-\alpha) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} : 0 \le \alpha \le 1 \right\}.$$

By comparing to the given $x(\mu)$, it follows that $x(\mu)$ is close to the optimal solution given by $\alpha = 0.6076$.

As the barrier trajectory avoids active constraints, $x(\mu)$ will converge to a basic feasible solution when $\mu \to 0$ only if the optimal solution is unique. This is not the case here.

4. (a) The dual objective $\varphi(u)$ is the optimal solution of

$$\begin{aligned} \varphi_1(u) = \min \ & -x_1 - 3x_3 - x_4 + u(4x_1 + 5x_2 + 6x_3 + 7x_4 - 10) \\ \text{s.t. } & x_1 + x_2 \le 1, \ x_3 + x_4 \le 1, \ x_j \in \{0,1\}, \ j = 1, \dots, 4, \\ = -10u - \max \ & (1-4u)x_1 - 5ux_2 + (3-6u)x_3 + (1-7u)x_4 \\ \text{s.t. } & x_1 + x_2 \le 1, \ x_3 + x_4 \le 1, \\ & x_j \in \{0,1\}, \ j = 1, \dots, 4. \end{aligned}$$

(b) For $u \in [0, 1/4]$, it is optimal to let $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ and $x_4 = 0$ in the Lagrangian relaxation problem. The dual objective function then becomes $\varphi(u) = -4$. Since φ is concave, it follows that it is maximized on a segment where it is constant.

(c) As the Lagrangian relaxed problem has integer extreme points if the integer requirement is relaxed, the bounds are equal. In fact, this case is very special in that they both give an optimal solution to (IP).

5. (a) Let $y_j = |x_j|$. Since the sign of x_j is irrelevant in the constraint of (P_1), and the term in the objective coefficient is $v_j x_j$, it follows that the sign of x_j is the negative sign of v_j in any optimal solution to (P_1). Consequently,

$$v_j x_j = v_j \operatorname{sign}(x_j)|x_j| = \operatorname{sign}(v_j)|v_j| \operatorname{sign}(x_j)|x_j| = -|v_j| y_j,$$

using $\operatorname{sign}(v_j)\operatorname{sign}(x_j) = -1$ whenever v_j and x_j are nonzero, plus $y_j = |x_j|$. Hence, (P_1) and (LP_1) are equivalent.

Let $y_k = 1$ for a k such that $|v_k| \ge |v_j|$, $j = 1, \dots, n$. The dual associated with (LP_1) takes the form

$$(DLP_1) \quad \begin{array}{ll} \text{maximize} & -u \\ \text{subject to} & u \ge |v_j|, \quad j = 1, \dots, n, \\ & u \ge 0. \end{array}$$

We then see that $y_k = 1$, $y_j = 0$, $j = 1, \ldots, k-1, k+1, \ldots, n$, and $u = |v_k|$ form a pair of primal and dual feasible solutions for which strong duality holds. Hence, they are optimal. We thus conclude that $x_k = -\operatorname{sign}(v_k)$, $x_j = 0$, $j = 1, \ldots, k-1, k+1, \ldots, n$, are optimal to (P_1). .... (4p)

(b) The suggested initial extreme points $v_1 = (1\ \ 0\ \ 0\ \ 0)^T$ and $v_2 = (0\ \ -1\ \ 0\ \ 0)^T$ give the initial basis matrix

$$B = \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (2\ 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{2}{5} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1\ c^T v_2) = (3\ -4)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ -\frac{13}{5} \end{pmatrix}.$$

By forming $c^T - y_1 A = (-13/5\ \ 13/5\ \ 4/5\ \ 6/5)$ we obtain the subproblem

$$\begin{array}{lll} \frac{13}{5} + \frac{1}{5} & \text{minimize} & -13x_1 + 13x_2 + 4x_3 + 6x_4 \\ & \text{subject to} & |x_1| + |x_2| + |x_3| + |x_4| \le 1. \end{array}$$

It follows from (5a) that v_1 and v_2 are optimal to the subproblem. Consequently, the optimal value of the subproblem is zero, and the original problem has been solved. The optimal solution x^* is given by

$$x^* = v_1\alpha_1 + v_2\alpha_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{3}{5} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} \frac{3}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \end{pmatrix}.$$