SF2812 Applied linear optimization, final exam
Wednesday June 10 2015 14.00–19.00
Brief solutions

1. (a) From the GAMS output file, the values of "`VAR x`" suggest $x = (2\ 0\ 0\ 3\ 1)$, the marginal costs for "`EQU cons`" suggest $y = (-1\ \ 2\ -2)^T$, and the marginal costs for "`VAR x`" suggest $s = (0\ 3\ 1\ 0\ 0)^T$. We have $Ax = b$, $A^Ty + s = c$, $x \geq 0$, $s \geq 0$ and $x^Ts = 0$. Hence, the solutions are optimal to the respective problem.

(b) Since $s_2 = 3$ and $s_3 = 1$, it follows that the optimal solution is unchanged if the costs of x_2 or x_3 are not decreased more than one unit. Hence, the solution is not at all sensitive to changes considered by AF. The computed optimal solution is optimal also considering the fluctuations.

(c) Since $y_3 = -2$, the optimal value is expected to change with -2 per unit change of b_3, i.e., if $b_5 = 5 + \delta$, the optimal value is expected to be $-12 - 2\delta$ for δ near zero.

2. (a) We get

$$\varphi(\lambda) = -\lambda b + \sum_{j=1}^{n} \min_{x_j \in \{0,1\}} (a_j\lambda - c_j)x_j = -\lambda b - \sum_{j=1}^{n} (c_j - \lambda a_j)_+,$$

(b) For a given λ, an optimal solution to the Lagrangian relaxed problem is $x(\lambda)$, with $x_j(\lambda) = 1$ for j such that $\lambda a_j - c_j < 0$ and $x_j(\lambda) = 0$ for j such that $\lambda a_j - c_j \geq 0$. A subgradient is now given by

$$-\left(\sum_{j=1}^{n} a_j x_j(\lambda) + b\right) = -b + \sum_{j: a_j\lambda < c_j} a_j.$$

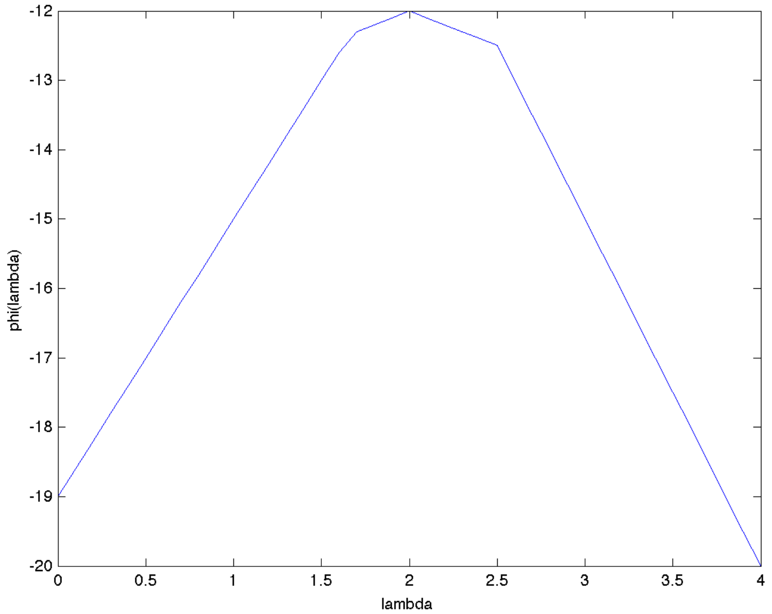
(c) We have $c_1/a_1 = 2$, $c_2/a_2 = 5/3$, $c_3/a_3 = 5/2$ so that

$$x(\lambda) = \begin{cases} (1\ 1\ 1)^T & \text{if } 0 \leq u \leq 5/3, \\ (1\ 0\ 1)^T & \text{if } 5/3 \leq u \leq 2, \\ (0\ 0\ 1)^T & \text{if } 2 \leq u \leq 5/2, \\ (0\ 0\ 0)^T & \text{if } u \geq 5/2. \end{cases}$$

Hence,

$$\varphi(\lambda) = \begin{cases} 4\lambda - 19 & \text{if } 0 \leq u \leq 5/3, \\ \lambda - 14 & \text{if } 5/3 \leq u \leq 2, \\ -\lambda - 10 & \text{if } 2 \leq u \leq 5/2, \\ -5\lambda & \text{if } u \geq 5/2. \end{cases}$$

The dual problem can be illustrated grahically as in the following figure.



The optimal solution is $\lambda^* = 2$ with $\varphi(\lambda^*) = -12$.

By inspection one can see that the optimal solution to (KP) is given by $x^* = (0\ 0\ 1)^T$ with $c^T x^* = -10$.

The duality gap is therefore 2.

3. (a) With $X = \operatorname{diag}(x)$ and $S = \operatorname{diag}(s)$, the linear system of equations takes the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = - \begin{pmatrix} Ax - b \\ A^T y + s - c \\ XSe - \mu e \end{pmatrix}.$$

Insertion of numerical values gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta y_1 \\ \Delta y_2 \\ \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \\ \Delta s_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ -3 \\ -5 \\ -5 \\ -3 \end{pmatrix}.$$

(b) If we compute $\alpha_{\max}$ as the largest step α for which $x + \alpha\Delta x \geq 0$ and $s + \alpha\Delta s \geq 0$, we obtain $\alpha_{\max} > 1$ so that we may for example let $\alpha = 1$. Then,

$$x^{(1)} \approx \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0.4588 \\ -0.7000 \\ 0.0235 \\ 0.2176 \end{pmatrix} \approx \begin{pmatrix} 4.4588 \\ 2.3000 \\ 2.0235 \\ 1.2176 \end{pmatrix},$$

$$y^{(1)} \approx \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1.5294 \\ 0.3353 \end{pmatrix} \approx \begin{pmatrix} 1.5294 \\ 0.3353 \end{pmatrix},$$

$$s^{(1)} \approx \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \approx \begin{pmatrix} -0.8647 \\ -1.2000 \\ -2.5353 \\ -3.8706 \end{pmatrix} \approx \begin{pmatrix} 0.1353 \\ 0.8000 \\ 0.4647 \\ 0.1294 \end{pmatrix}.$$

(The numerical values of $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ are not required.)

4. (See the course material.)

5. (a) For the given cut patterns, we obtain

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_B = B^{-1}b = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix}, \quad y = B^{-T}e = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $y \geq 0$ no slack variables enters the basis.
The subproblem is given by

$$\begin{array}{lll} 1 - & \frac{1}{6}\text{maximize} & 2\alpha_1 + 3\alpha_2 + 6\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \leq 11, \\ & & \alpha_i \geq 0, \text{ integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is $\alpha^* = (2\ 1\ 0)^T$ with optimal value $-1/6$. Hence, $a_4 = (2\ 1\ 0)^T$ and the maximum step is given by

$$0 \leq x = B^{-1}b - \eta B^{-1}a_4 = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix} - \eta \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Hence, $\eta_{\max} = 30$ and x_1 leaves the basis, so that the basic variables are given by $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$. The reduced costs are given by

$$y = B^{-T}e = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which gives $y_1 = 1/4$, $y_2 = 1/2$ and $y_3 = 1$.
The subproblem is given by

$$
\begin{array}{rll}
1 - \frac{1}{4}\text{maximize} & \alpha_1 + 2\alpha_2 + 4\alpha_3 \\
\text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \le 11, \\
 & \alpha_i \ge 0, \ \text{integer}, \quad i = 1, 2, 3.
\end{array}
$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value is zero, so that the linear program has been solved. The optimal solution is $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$, with $a_2 = (0\ 2\ 0)^T$, $a_3 = (0\ 0\ 1)^T$ and $a_4 = (2\ 1\ 0)^T$.

(b) The solution given by the linear programming relaxation happens to be integer valued. This means that we have solved the original problem as well. The optimal solution is to use 80 W-rolls, with 10 rolls cut according to pattern $(0\ 2\ 0)^T$, 40 rolls cut according to pattern $(0\ 0\ 1)^T$ and 30 rolls cut according to pattern $(2\ 1\ 0)^T$.

(Note that this is very special. In general one can not expect to obtain an optimal integer solution in this way.)