SF2812 Applied linear optimization, final exam
Wednesday June 8 2016 8.00–13.00
Brief solutions

1. (a) There is at least one optimal solution, which is integer valued. However, if the optimal solution is nonunique, there will also be noninteger optimal solutions.

(b) Since $\widehat{X}$ is nonnegative, summation of rows and columns of $\widehat{X}$ shows that $\widehat{X}$ is feasible. If we let the matrix S denote the dual slacks, i.e., $s_{ij} = c_{ij} - \widehat{u}_i - \widehat{v}_j$, then

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Consequently, S has nonnegative components. In addition, complementarity holds, since $\widehat{x}_{ij} s_{ij} = 0$, $i = 1, 2$, $j = 1, 2, 3$. This means that we have optimal solutions to the two problems.

(c) The nonzero components of the given W correspond to strictly positive components of $\widehat{X}$. Since W has row sum as well as column sum zero, it follows that $\widehat{X} + \alpha W$ is optimal as long as $\widehat{X} + \alpha W$ is nonnegative. The most limiting positive and negative values of α are -0.5 and 1.5 respectively. These values correspond to two integer valued optimal solutions:

$$\widehat{X} - 0.5W = \begin{pmatrix} 6 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \widehat{X} + 1.5W = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 5 & 0 \end{pmatrix}.$$

(d) Since $\widehat{X}$ is not an extreme point, it is not provided as a solution by the simplex method.

2. (See the course material.)

3. (a) With $X = \text{diag}(x)$ and $S = \text{diag}(s)$, the linear system of equations takes the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = - \begin{pmatrix} Ax - b \\ A^T y + s - c \\ XSe - \mu e \end{pmatrix},$$

for a suitable value of the barrier parameter μ. We may for example let $\mu = x^T s/n = 5$. Insertion of numerical values gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta y_1 \\ \Delta y_2 \\ \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \\ \Delta s_4 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ -1 \\ -1 \\ -1 \\ -3 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

(b) We would compute $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ as $x^{(1)} = x^{(0)} + \alpha\Delta x^{(0)}$, $y^{(1)} = y^{(0)} + \alpha\Delta y^{(0)}$, $s^{(1)} = s^{(0)} + \alpha\Delta s^{(0)}$, where α is a positive steplength. In a pure Newton step, $\alpha = 1$, but we must also maintain $x^{(1)} > 0$ and $s^{(1)} > 0$. We may compute $\alpha_{\max}$ as the largest step α for which $x + \alpha\Delta x \geq 0$ and $s + \alpha\Delta s \geq 0$. We may then let $\alpha = \min\{1, 0.99\alpha_{\max}\}$ to ensure positivity of $x^{(1)} > 0$ and $s^{(1)} > 0$. (In order to get a convergent method, some additional condition on α ensuring proximity to the barrier trajectory may need to be imposed.)

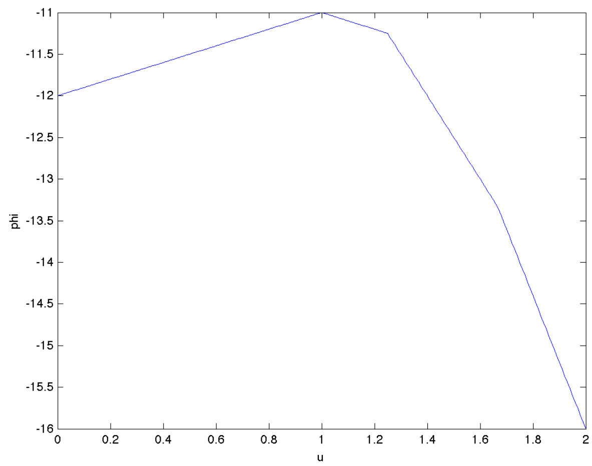
4. (a) For a given nonnegative u, the resulting Lagrangian relaxed problem gives the dual objective function $\varphi(u)$ as

$$\begin{array}{rll} \varphi(u) = -8u + & \text{minimize} & (3u-5)x_1 + (6u-7)x_2 + (7u-10)x_3 \\ & \text{subject to} & -x_1 - 2x_2 - 3x_3 \geq -3, \\ & & x_j \in \{0,1\}, \quad j = 1,\dots,n. \end{array}$$

There are only five feasible solutions to the relaxed problem, $(0\ 0\ 0)^T$, $(1\ 0\ 0)^T$, $(0\ 1\ 0)^T$, $(0\ 0\ 1)^T$ and $(1\ 1\ 0)^T$. By enumerating these solutions, we obtain

$$\varphi(u) = \min\{-8u, -5u-5, -2u-7, -u-10, u-12\}.$$

The dual problem may be illustrated graphically as:



It can be seen that the optimal solution is 1 and the optimal value is -11.

(b) Since the Lagrangian dual gives a relaxation whose bound is always at least as good as the linear programming relaxation, the optimal value of the linear programming relaxation problem cannot be greater than -11.

5. (a) For the given cut patterns, we obtain

$$B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_B = B^{-1}b = \begin{pmatrix} 15 \\ 25 \\ 50 \end{pmatrix}, \quad y = B^{-T}e = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $y \geq 0$ no slack variables enters the basis.

The subproblem is given by

$$\begin{array}{rll} 1 - & \frac{1}{4}\text{maximize} & \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \leq 12, \\ & & \alpha_i \geq 0, \ \text{integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is $\alpha^* = (1\ 0\ 1)^T$ with optimal value $-1/4$. Hence, $a_4 = (1\ 0\ 1)^T$ and the maximum step is given by

$$0 \leq x = B^{-1}b - \eta B^{-1} a_4 = \begin{pmatrix} 15 \\ 25 \\ 50 \end{pmatrix} - \eta \begin{pmatrix} \frac{1}{4} \\ 0 \\ 1 \end{pmatrix}.$$

Hence, $\eta_{\max} = 50$ and x_3 leaves the basis, so that the basic variables are given by $x_1 = 5/2$, $x_2 = 25$ and $x_4 = 50$. The reduced costs are given by

$$y = B^{-T} e = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which gives $y_1 = 1/4$, $y_2 = 1/2$ and $y_3 = 3/4$.
The subproblem is given by

$$\begin{array}{rll} 1 - & \frac{1}{4}\text{maximize} & \alpha_1 + 2\alpha_2 + 3\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \leq 12, \\ & & \alpha_i \geq 0, \ \text{integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value is zero, so that the linear program has been solved. The optimal solution is $x_1 = 5/2$, $x_2 = 25$ and $x_4 = 50$, with $a_1 = (4\ 0\ 0)^T$, $a_2 = (0\ 2\ 0)^T$ and $a_4 = (1\ 0\ 1)^T$.

(b) The solution given by the linear programming relaxation may be rounded up to give a feasible solution $\widetilde{x}$ to the original problem. In this case, $\widetilde{x}_1 = 3$, $\widetilde{x}_2 = 25$ and $\widetilde{x}_4 = 50$. This gives a total of 78 W-rolls. The linear programming relaxation gives 77.5 W-rolls, which is a lower bound for the original problem. Since the number of W-rolls is integer valued, we conclude that 78 is a lower bound, so that $\widetilde{x}$ in fact is an optimal solution to the original problem. The optimal solution is therefore to use 78 W-rolls, with 3 rolls cut according to pattern $(4\ 0\ 0)^T$, 25 rolls cut according to pattern $(0\ 2\ 0)^T$ and 50 rolls cut according to pattern $(1\ 0\ 1)^T$.

(Note that this is very special. In general one can not expect to obtain an optimal integer solution in this way.)