KTH Mathematics

SF2812 Applied linear optimization, final exam
Friday January 9 2009 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser; plus a calculator provided by the department.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear program (LP) defined as

$$(LP) \qquad \begin{array}{ll} \min & x_1 + x_2 \\ \text{då} & x_1 + x_2 = 1, \\ & x_1 \ge 0,\ x_2 \ge 0. \end{array}$$

(a) For a fixed positive barrier parameter μ, formulate the primal-dual system of nonlinear equations corresponding to the problem above. Use the fact that the problem is small to give explicit expressions for the solution $x(\mu)$, $y(\mu)$ and $s(\mu)$ to the system of nonlinear equations. .... (6p)

(b) Calculate optimal solutions to (LP) and the corresponding dual problem by letting $\mu \to 0$ in the expressions given in (1a). Verify optimality. .... (2p)

(c) If the simplex method had been used for solving (LP), could the optimal solution to (LP) given in (1b) have been obtained? .... (2p)

2. Consider the linear program (LP) defined by

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \ge 0, \end{array}$$

where A is a given $m \times n$-matrix with linearly independent rows. Let $S = \{x : Ax = b, x \ge 0\}$.

(a) Define a convex set. .... (2p)

(b) Show that S is a convex set. .... (2p)

(c) Define a basic feasible solution to (LP). .... (3p)

(d) Show that x is an extreme point to S if and only if x is a basic feasible solution to (LP). .... (3p)

3. Consider the linear integer programming problem (IP) defined as

$$(IP) \qquad \begin{array}{ll} \text{minimize} & -6x_1 - 10x_2 - 11x_3 - 10x_4 \\ \text{subject to} & -x_1 - x_2 - x_3 \geq -2, \\ & -x_2 - x_3 - x_4 \geq -2, \\ & -3x_1 - 4x_2 - 5x_3 - 6x_4 \geq -9, \\ & x \geq 0, \ x \text{ integer}. \end{array}$$

(a) Assume that the two first constraints are relaxed by Lagrangian relaxation with corresponding multipliers $u_1 = 1$ and $u_2 = 1$. Formulate the Lagrangian relaxed problem. Find an optimal solution to the Lagrangian relaxed problem by making use of the fact that the knapsack problem

$$\begin{array}{ll} \text{maximize} & 5x_1 + 8x_2 + 9x_3 + 9x_4 \\ \text{subject to} & 3x_1 + 4x_2 + 5x_3 + 6x_4 \leq 9, \\ & x \geq 0, \ x \text{ integer}, \end{array}$$

has optimal solution $(0\ 1\ 1\ 0)^T$. .... (4p)

(b) Find a subgradient to the dual objective function corresponding to the Lagrangian relaxation in exercise (3a) at the point $u = (1\ 1)^T$. Your subgradient has a special property which makes it possible to obtain an optimal solution to (IP). Which one? Use this property to find an optimal solution to (IP)..(6p)

4. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 & 1 & -1 & c_4 \end{pmatrix}^T.$$

Are there values of b_1, b_2, c_1 and c_4 such that $\widehat{x} = (3\ 2\ 1\ 0)^T$ is optimal to (LP)? If so, determine all such values. .... (10p)

Hint: It holds that $Av = 0$ for $v = (1\ \ -2\ \ 1\ \ 0)^T$.

5. Consider a cutting-stock problem with the following data:

$$W = 11, \quad m = 3, \quad w_1 = 3, \quad w_2 = 4, \quad w_3 = 5, \quad b = \begin{pmatrix} 20 & 15 & 10 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook. Given are rolls of width W. Rolls of m different widths are demanded. Roll i has width w_i, $i = 1, \ldots, m$. The demand for roll i is given by b_i, $i = 1, \ldots, m$. The aim is to cut the W-rolls so that a minimum number of W-rolls are used.

Cut patterns $(0\ 0\ 2)^T$, $(1\ 2\ 0)^T$ and $(2\ 0\ 1)^T$ have been suggested.

(a) Consider the LP-relaxed problem associated with the above problem. Determine a basic feasible solution associated with the above three cut patterns. Show that this basic feasible solution is optimal to the LP-relaxed problem. The subproblem may be solved in any way, that need not be systematic. . (7p)

(b) Determine a "near-optimal" solution to the original problem. Give a bound on the maximum deviation from the optimal value of the original problem. . . (3p)

Good luck!