KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday October 20 2011 14.00–19.00

Examiner: Anders Forsgren.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear programming problem (PLP) and its dual (DLP) defined as

$$(PLP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 3 & 1 & 0 & 0 \end{pmatrix}^T.$$

AF has implemented a primal-dual interior method in Matlab. He has tried to solve the above linear program by his solver and obtained the following approximate numbers for x, y, and s:

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x' =  3.0000   -0.0000    0.0000    2.0000

y' =  1.5000    0.0000

s' =  0.0000   -2.0000    1.5000    0.0000
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AF is certain that he has entered the problem data correctly, and that his initial estimates of x and s were strictly positive vectors. However, he is a bit confused by the result.

(a) Which properties of the approximate solution should make AF suspicious? (2p)

(b) Based on the conditions that his approximate solution fulfills, what mistake do you think AF has made in his implementation? .... (2p)

(c) Which problem has AF in fact computed an approximate solution to? ...(2p)

(d) Solve (PLP) by the simplex method. Start with the basic feasible solution that results by rounding the value of x that AF has computed to the nearest integers. ......(4p)

2. Consider the stochastic program (P) given by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & T(\omega)x = h(\omega), \\ & x \geq 0, \end{array}$$

where ω is a stochastic variable and $T(\omega)x = h(\omega)$ is to be interpreted as an "informal" stochastic constraint. Assume that ω takes on a finite number of values $\omega_1, \ldots, \omega_N$ with corresponding probabilities $p_1, \ldots, p_N$. Let T_i denote $T(\omega_i)$ and let h_i denote $h(\omega_i)$.

(a) Explain how the deterministically equivalent problem

$$\begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i q_i^T y_i \\ \text{subject to} & Ax = b, \\ & T_i x + W y_i = h_i, \quad i = 1, \ldots, N, \\ & x \geq 0, \\ & y_i \geq 0, \quad i = 1, \ldots, N, \end{array}$$

arises. (We assume, for simplicity, "fix compensation", i.e., W does not depend on i.) ......(6p)

(b) Define *VSS* in terms of suitable optimization problems. ......(2p)

(c) Define *EVPI* in terms of suitable optimization problems. ......(2p)

3. Consider the linear program (LP) given by

$$(LP) \quad \begin{array}{ll} \text{minimize} & 2x_1 + 2x_2 + x_4 \\ \text{subject to} & 2x_1 + x_2 - x_3 + 2x_4 = 2, \\ & -1 \leq x_j \leq 1, \quad j = 1, \ldots, 4. \end{array}$$

Solve (LP) by Dantzig-Wolfe decomposition. Consider $2x_1 + x_x - x_3 + 2x_4 = 2$ the complicating constraint, and consider $-1 \leq x_j \leq 1$, $j = 1, \ldots, 4$, the easy constraints.

Use the extreme points $v_1 = (-1 \ \ -1 \ \ -1 \ \ 1)^T$ and $v_2 = (1 \ \ -1 \ \ -1 \ \ 1)^T$ for obtaining an initial feasible solution to the master problem.

The subproblem(s) that arise may be solved in any way, that need not be systematic. ......(10p)

4. Consider the integer program (IP) defined as

$$
(IP) \qquad \begin{array}{ll} \text{minimize} & -2x_1 - x_2 - 3x_3 - x_4 \\ \text{subject to} & -4x_1 - 5x_2 - 6x_3 - 7x_4 \geq -10, \\ & -x_1 - x_3 \geq -1, \\ & x_j \in \{0,1\}, \quad j = 1, \dots, 4. \end{array}
$$

Assume that the constraint $-x_1 - x_3 \geq -1$ is relaxed with corresponding nonnegative multiplier u. Let $\varphi(u)$ denote the resulting dual objective function.

In this exercise, you may make use of the fact that the dual problem is one-dimensional, and that the the Lagrangian relaxation problem is of small dimension. The methods used need not be systematic.

(a) Formulate the Lagragian relaxation problem whose optimal value gives $\varphi(u)$. .... (3p)

(b) Give an explicit expression of $\varphi(u)$ for $u \geq 0$ and show that the optimal solution to the dual problem is given by $u^* = 2$. You may for example enumerate all feasible solutions to the Lagrangian relaxation problem and make use of them. .... (4p)

(c) Give two optimal solutions to the Lagrangian relaxation problem where $u = u^* = 2$. Finally, use these solutions to give two subgradients to φ at u^*... (3p)

5. Consider the linear programming problem (PLP) and its dual (DLP) defined as

$$
(PLP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}
$$

Assume that $c \geq 0$.

In the discussion below, we let $\text{optval}(PLP) = \infty$ if (PLP) is infeasible and analogously $\text{optval}(DLP) = -\infty$ if (DLP) is infeasible, where "optval" denotes the optimal value.

(a) Show that (DLP) always has a feasible solution. Use this fact to give a lower bound on the optimal value of (DLP). .... (2p)

(b) Give a lower bound on the optimal value of (PLP) with arguments based on (PLP) only. Is (PLP) necessarily feasible? .... (2p)

(c) Assume that there exists $\eta \in I\!R^m$ such that $A^T \eta \leq 0$ and $b^T \eta > 0$. What is the implication on (PLP) and (DLP)? .... (3p)

(d) Assume that there exists no $\eta \in I\!R^m$ such that $A^T \eta \leq 0$ and $b^T \eta > 0$. What is the implication on (PLP) and (DLP)? .... (3p)

Good luck!