KTH Mathematics

SF2812 Applied linear optimization, final exam
Saturday February 18 2012 9.00–14.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Given a linear program (LP_δ) defined for a scalar δ,

$$(LP_\delta) \qquad \begin{array}{ll} \min & c^T x \\ \text{då} & Ax = b + \delta e_3, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$c = \begin{pmatrix} 4 & 13 & 11 & 0 & 0 & 0 \end{pmatrix}^T.$$

(a) An optimal basic solution for (LP_0) is given by $x = (1 \;\; 2 \;\; 3 \;\; 0 \;\; 0 \;\; 0)^T$ with corresponding optimal dual solution $y = (4 \;\; 5 \;\; 6)^T$ and $s = (0 \;\; 0 \;\; 0 \;\; 4 \;\; 5 \;\; 6)^T$. Use this information to determine an underestimate of the optimal value of (LP_δ) on the form $\alpha + \beta\delta$. The underestimate should be valid for any δ for which (LP_δ) has feasible solutions, and it should be exact in a neighborhood of $\delta = 0$. Your task is thus to determine suitable values of α and β. . . . .(6p)

(b) Determine limits on δ such that the underestimate $\alpha + \beta\delta$ is exact within these limits, for the values of α and β which you calculated in (1a). . . . .(4p)

2. Let (P) and (D) be defined by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

(a) Let x be a feasible solution to (P) and let y, s be a feasible solution to (D). Show that the duality gap for these solutions is given by $x^T s$ and motivate the conclusion that we have optimal solutions for the two problems if and only if $x_j \cdot s_j = 0$ for all j. . . . . (4p)
(It may be assumed known that if (P) has an optimal solution, then (D) has an optimal solution, and the optimal values are equal.)

(b) Show that if (P) has an optimal solution, then there is at least one extreme point (basic feasible solution) which is optimal. . . . . (6p)
(You may for example use the representation theorem without proof.)

3. Consider the optimization problem

$$
(IP) \qquad \begin{array}{ll}
\text{minimize} & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{j=1}^{n} f_j z_j \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,\dots,n, \\
& \displaystyle\sum_{i=1}^{n} a_i x_{ij} \ge b_j z_j, \quad j = 1,\dots,n, \\
& x_{ij} \in \{0,1\},\ z_j \in \{0,1\}, \quad i = 1,\dots,n,\ j = 1,\dots,n,
\end{array}
$$

where a_i, $i = 1,\dots,n$, b_j, $j = 1,\dots,n$, c_{ij}, $i = 1,\dots,n$, $j = 1,\dots,n$, and f_j, $j = 1,\dots,n$, are integer nonnegative constants.

(a) Let $\varphi(u)$ denote the dual objective function that results when the constraints

$$\sum_{j=1}^{n} x_{ij} - 1 = 0, \quad i = 1,\dots,n,$$

are relaxed by Lagrangian relaxation for multipliers u_i, $i = 1,\dots,n$. Simplify the problem that gives $\varphi(u)$ as much as you can and discuss the structure of this problem. . . . . (6p)

(b) An alternative would be to form a dual problem by Lagrangian relaxation of the constraints

$$\sum_{i=1}^{n} a_i x_{ij} - b_j z_j \ge 0,, \quad j = 1,\dots,n,$$

instead. Comment on the quality of the lower bound on the optimal value of (IP) given by the resulting dual problem, compared to bound on the optimal value of (IP) given by the dual problem formulated in (3a). . . . . (4p)

4. Consider a cutting-stock problem with the following data:

$$W = 14, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 7, \quad b = \begin{pmatrix} 40 & 90 & 50 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook. Given are rolls of width W. Rolls of m different widths are demanded. Roll i has width w_i,

$i = 1, \dots, m$. The demand for roll i is given by b_i, $i = 1, \dots, m$. The aim is to cut the W-rolls so that a minimum number of W-rolls are used.

Solve the LP-relaxed problem associated with the above problem. Use the pure cut patterns to create an initial basic feasible solution, i.e., create one cut pattern with only w_1-rolls and correspondingly for w_2 and w_3.

You may utilize the fact that the subproblems that arise are small, and they may be solved in any way, that need not be systematic. We suggest that you do not use dynamic programming but instead solve the subproblem by enumeration and in case of non-unique solution selects the one with the most w_2-rolls. (As the requirement for w_2-rolls is the significantly largest.)

Finally create a “good” solution to the original problem based on your solution to the LP-relaxed problem. Comment on the quality of this solution. .... (10p)

5. Let $P = \{x \in I\!R^n : Ax \geq b\}$, where $A \in I\!R^{m \times n}$ and $b \in I\!R^m$. We say that an extreme point of P is *degenerate* if there are more than n active constraints at the extreme point. If a_i^T denotes the ith row of A and b_i denotes the ith component of b, we say that a constraint $a_k^T x \geq b_k$ is *redundant* if the constraint is not needed to describe P, i.e., if $P = \{x \in I\!R^n : a_i^T x \geq b_i, i = 1, \dots, k-1, k+1, \dots, m\}$.

One could think that a degenerate extreme point implies the existence of a redundant constraint. Your task here is to demonstrate that this need not be the case.

For the remainder of this exercise, let $P = \{x \in I\!R^3 : Ax \geq b\}$, where

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & -2 \\ -1 & 1 & -2 \\ -1 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ -2 \\ -2 \\ -2 \\ 0 \end{pmatrix}.$$

When considering this P, you need not use systematic methods, and you may utilize the fact that the problem is of low dimension.

(a) The point $\bar{x} = (0\ 0\ 1)^T$ is an extreme point of P. (This need not be verified.) Show that $\bar{x}$ is a degenerate extreme point of P. .... (2p)

(b) Show that P as described by A and b contains no redundant constraints. . (5p)
Hint: Assume that the first constraint is redundant. Consider

$$x(t) = \begin{pmatrix} -t & -t & 1 \end{pmatrix},$$

for $t > 0$. Which constraints does $x(t)$ satisfy? Make analogous arguments for the other constraints.

(c) Explain why a straightforward convergence proof of the primal simplex method for linear programming would fail if primal nondegeneracy is not assumed. Primal degeneracy is here to be interpreted as existence of a degenerate extreme point in the primal linear program. .... (3p)

Good luck!